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β-VISCOSITY SOLUTIONS OF HAMILTON-JACOBI EQUATIONS AND APPLICATIONS TO A CLASS OF OPTIMAL CONTROL PROBLEMS

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INTRODUCTION

First-order Hamilton-Jacobi equations (HJEs) comprise an important class of nonlinear partial differential equations (PDEs) with many applications. Typical examples can be found in mechanics, optimal control theory, etc. Specifically, this class includes dynamic programming equations arising in deterministic optimal controls, which are known as Hamilton-Jacobi-Bellman equations. In general, these nonlinear equations do not have classical solutions. As a result, it is necessary to study weak solutions and a viscosity solution is such a week solution.

The theory of viscosity solutions for partial differential equations appeared in 1980s. In particular, in the paper by M. G. Crandall and P. L. Lions (1983), the authors introduced the viscosity solution as a generalized solution of partial differential equations. Instead of requiring that the solution u satisfies the given equation almost everywhere, it is sufficient for u to be a continuous functions satisfying a pair of inequalities via sufficiently smooth test functions, or via subdifferential and superdifferential.

The viscosity solution is an effective device to study nonlinear Hamilton-Jacobi equations. We emphasize that a viscosity solution is a weak solution since it is only continuous and its derivative is defined through test functions and the extremal principle. However, it has been proved that viscosity solution can be defined by subdifferential, superdifferential, which are called semiderivatives. It leads to a tight connection between the theory of viscosity solution and nonsmooth analysis which includes subdifferential theory.

Since 1993, the smooth variational principle, which was proved by Deville, has been widely employed as an important tool to establish the uniqueness of β-viscosity solution, in the class of continuous and bounded functions, of Hamilton-Jacobi equations of the form $u + F(Du) = f$, where F is uniformly continuous on X^*_β and f is uniformly continuous and bounded on X.

Optimal control problems were introduced in 1950s. It is well known that they have many applications in Mathematics, Physics, and application areas. By the dynamic programming principle, the value function of an optimal control problem is a solution to an associated partial differential equation. Unfortunately, since value functions might not be differentiable, several approaches have been introduced to study them. The viscosity solution again is an effective approach to investigate optimal control theory. To the best of our knowledge, treating optimal control problems by viscosity solutions via subdiffrential is scared especially if the value function is unbounded.

Recently, an increasing literature has been devoted to to the study of HamiltonJacobi equation on junctions and networks. The authors established properties of the value function, the comparison principle for optimal control problems with bounded running cost l. Although many important results have been obtained, it seems that the assumptions in the recent work are quite strict.

We focus on β-subdifferential, the uniqueness of β-viscosity solution for Hamilton-Jacobi equations of the forms $u+H(x,Du)=0$ and $u+H(x,u,Du)=0$, the existence and stability of β-viscosity solution. Moreover, there are many applications of β-viscosity solutions for optimal control problems. Motivated by that fact, we are also interested in finding necessary and sufficient conditions for optimal control problems in infinite dimensional spaces. The new approach of viscosity solution on junctions is another topic of our interest. Based on the known model of classical viscosity solution, the uniqueness and applications of viscosity solutions for optimal control problems on junctions are promising topics.

In addition to Introduction, Conclusion, and References List, the dissertation consists of four chapters.

In Chapter 1, we present the notion of β-viscosity solution and its properties, and several results on the smooth variational principle.

in Chapter 2, We prove the uniqueness of β-viscosity solution for Hamilton-Jacobi equations of the general form $u+H(x,u,Du)=0$ in Banach spaces. The stability and existence of the solution of such equations are also investigated.

In Chapter 3, we show that the value function of a certain optimal control problem is a β-viscosity solution of the associated Hamilton-Jacobi equation. The feedback controls and also sufficient conditions for optimality are also studied in this chapter.

In Chapter 4, we present the notion of junctions, assumptions and optimal control problems on junctions. Several properties of the value function such as the continuity on $\mathcal{G}$, the local Lipschitz at O on each J_i, estimates of the value function at O through Hamilton. We prove that the value function of an optimal control problem on junctions is a viscosity solution of the associated Hamilton-Jacobi equation. We also apply our results in such optimal control problem.

Chapter 1
β-SUBDERIVATIVE

In this chapter, we present β-viscosity subdifferential on Banach space X and prove the smooth β-variational principle which will be used to establish the uniquess of β-viscosity solution.

1.1. β-differentiable

Definition 1.1.1. *A* borno β *on* X *is a family of closed, bounded, and centrally symmetric subsets of* X *satisfying the following three conditions:*

1) $X = \bigcup_{B\in\beta} B$;

2) β *is closed under scalar multiplication,*

3) the union of any two elements in β *is contained in some element of* β.

By Theorem 27 in [Hoang Tuy, 2005], a borno β in Definition 1.1.1 defines on X^* a locally convex Hausdorff topology τ_β. The space X^* with this topology τ_β is denoted by X^*_β. A local base of the origin 0 in X^*_β is the collection of sets of the form

$$\{f : |f(x)| < \varepsilon, \quad \forall x \in M\},$$

where $\epsilon > 0$ is arbitrry and $M \in \beta$.

Then, the sequence $(f_m) \subset X^*$, converges to $f \in X^*$ with respect to τ_β if and only if for any $M \in \beta$ and any $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $|f_m(x) - f(x)| < \varepsilon$ for all $m \geq n_0$ and $x \in M$; that is, f_m converges uniformly to f on M. Hence τ_β is also called *the uniformly convergent topology on elements of* β.

Example 1.1.2. *It is easy to verify the following facts. 1) The family* F *of all closed, bounded, and centrally symmetric subsets of* X *is a borno on* X, *which is called Fréchet borno.*
2) The family H *of all compact, centrally symmetric subsets of* X *is a borno on* X *called Hadamard borno.*
3) The family WH *of all weakly compact, closed, and centrally symmetric subsets of* X *is a borno on* X *called weak Hadamard borno.*

4) The family G of all finite, centrally symmetric subsets of X is also a borno on X called Gâteaux borno.

Remark 1.1.3. *If β borno is F (Fréchet), H (Hadamard), WH (Hadamard weak) or G (Gâteaux), then we have Fréchet topology, Hadamard topology, Hadamard weak topology and Gâteaux topology on the dual space X^*, respectively. Thus, F-topology is the strongest topology and G topology is the weakest topology among β-topologies on X^*.*

Definition 1.1.4. *Given a function $f : X \to \overline{\mathbb{R}}$. We say that f is β-differentiable at $x_0 \in X$ with β-derivative $\nabla_\beta f(x_0) = p \in X^*$ if $f(x_0) \in \mathbb{R}$ and*

$$\lim_{t\to 0} \frac{f(x_0 + th) - f(x_0) - \langle p, th\rangle}{t} = 0$$

uniformly in $h \in V$ for every $V \in \beta$.

We say that the function f is β-smooth at x_0 if there exists a neighborhood U of x_0 such that f is β-differentiable on U and $\nabla_\beta f : U \to X^*_\beta$ is continuous.

1.2. β-viscosity subdifferential

Definition 1.2.1. *Let $f : X \to \overline{\mathbb{R}}$ be a lower semicontinuous function and $f(x) < +\infty$. We say that f is β-viscosity subdifferentiable and x^* is a β-viscosity subderivative of f at x if there exists a local Lipschitzian function $g : X \to \mathbb{R}$ such that g is β-smooth at x, $\nabla_\beta g(x) = x^*$ and $f - g$ attains a local minimum at x. We denote the set of all β-subderivatives of f at x by $D^-_\beta f(x)$, which is called β-viscosity subdifferential of f at x.*

Let $f : X \to \overline{\mathbb{R}}$ be an upper semicontinuous function and $f(x) > -\infty$. We say that f is β-viscosity superdifferentiable and x^* is a β-viscosity superderivative of f at x if there exists a local Lipschitzian function $g : X \to \mathbb{R}$ such that g is β-smooth at x, $\nabla_\beta g(x) = x^*$ and $f - g$ attains a local maximum at x. We denote the set of all β-superderivatives of f at x by $D^+_\beta f(x)$, which is called β-viscosity superdifferential of f at x.

Theorem 1.2.2. *1) If $\beta_1 \subset \beta_2$ then $D^-_{\beta_2} f(x) \subset D^-_{\beta_1} f(x)$; in particular, $D^-_F f(x) \subset D^-_\beta f(x) \subset D^-_G f(x)$ for every borno β.*

2) If f is continuous, $f(x)$ is finite and $D^-_\beta f(x)$, $D^+_\beta f(x)$ are two nonempty sets, then f is β-differentiable at x.
3) If $\beta_1 \subset \beta_2$ and f is β_1-differentiable at x and f is β_2-viscosity subdifferentiable at x, then $D^-_{\beta_2} f(x) = \{\nabla_{\beta_1} f(x)\}$.
4) $D^-_\beta f(x) + D^-_\beta g(x) \subset D^-_\beta (f+g)(x)$.
5) $D^-_\beta f(x)$ is a convex set.

We have the following results.

Remark 1.2.3.
1) $D^-_F f(x) \subset D^-_{WH} f(x) \subset D^-_H f(x) \subset D^-_G f(x)$.
2) If X is a reflexive space, then $D^-_F f(x) = D^-_{WH} f(x)$.
3) If $X = \mathbb{R}^n$, then $D^-_F f(x) = D^-_{WH} f(x) = D^-_H f(x)$.
4) If $X = \mathbb{R}$ then $D^-_F f(x) = D^-_G f(x)$.

Theorem 1.2.4. *If f is a convex function defined on the convex set C and $x \in C$, then for every borno β we have*

$$D^-_\beta f(x) = D^-_G f(x) = \partial f(x).$$

Next, we denote

$$\mathcal{D}_\beta(X) = \{g : X \to \mathbb{R} \,|\, g \text{ is bounded, Lipschitzian, and } \beta\text{-differentiable on } X\},$$

$$\|g\|_\infty = \sup\{|g(x)| : \ x \in X\}, \quad \|\nabla_\beta g\|_\infty = \sup\{\|\nabla_\beta g(x)\| : \ x \in X\}$$

and

$$\mathcal{D}^*_\beta(X) = \{g \in \mathcal{D}_\beta(X) | \ \nabla_\beta g : X \to X^*_\beta \text{ is continuous}\}.$$

The following hypotheses will be used in the derivation of our results.
(H_β) There exists a bump function b such that $b \in \mathcal{D}_\beta(X)$; and
(H^*_β) There exists a bump function b (i.e. its support is nonempty and bounded) such that $b \in \mathcal{D}^*_\beta(X)$.

Proposition 1.2.5. *The hypotheses (H_β) and (H^*_β) are fulfilled if the Banach space X has a β-smooth norm.*

Proposition 1.2.6. *Let X be a Banach space satisfying (H_β) (resp. (H^*_β)) and E a closed subset of X. Then, for a lower semicontinuous bounded from below function f on E and any $\varepsilon \in (0,1)$, there exist a $g \in \mathcal{D}_\beta(X)$ (resp. $g \in \mathcal{D}^*_\beta(X)$) and an $x_0 \in E$ such that:*

(a) $f+g$ attains its minimum at x_0.

(b) $\|g\|_\infty \le \varepsilon$ and $\|\nabla_\beta g\|_\infty \le \varepsilon$.

Proposition 1.2.7. *Assuming the real Banach space X satisfying hypothesis (H^*_β) and u, v are two bounded functions on X such that u is upper semicontinuous and v is lower semicontinuous. Then, there exists a constant C such that for every $\varepsilon \in (0,1)$, there are $x, y \in X, p \in D^+_\beta u(x), q \in D^-_\beta v(y)$ such that:*

(a) $\|x-y\| < \varepsilon^2$ and $\|p-q\| < \varepsilon$;

(b) For every $z \in X$, $v(z) - u(z) \ge v(y) - u(x) - \varepsilon$;

(c) $\|x-y\|\sqrt{\|p\|} < C\varepsilon, \quad \|x-y\|\sqrt{\|q\|} < C\varepsilon$.

Theorem 1.2.8. *Let X be a Banach space with an equivalent β-smooth norm and $f_1, \cdots, f_N : X \to \overline{\mathbb{R}}$ be N lowwer semicontinuous bounded from below functions and*

$$\lim_{\eta\to 0} \inf\{\sum_{n=1}^{N} f_n(y_n) : diam(y_1, \cdots, y_N) \le \eta\} < +\infty.$$

Then, for any $\varepsilon > 0$, there exist $x_n \in X, n = 1, \cdots, N$ and $x^*_n \in D^-_\beta f_n(x_n)$ satisfying

$$(i)\ diam(x_1, \cdots, x_N) \max(1, \|x^*_1\|, \cdots, \|x^*_N\|) < \varepsilon;$$

$$(ii)\ \sum_{n=1}^{N} f_n(x_n) < \inf_{x\in X} \sum_{n=1}^{N} f_n(x) + \varepsilon;$$

$$(iii)\ \left\| \sum_{n=1}^{N} x^*_n \right\| < \varepsilon.$$

Theorem 1.2.9. *Let X be a Banach space with an equivalent β-smooth norm, Ω an open subset of X, and $f_1, \cdots, f_N : \overline{\Omega} \to \mathbb{R}$ are N lowwer semicontinuous bounded from below functions. Then, for any $\varepsilon > 0$, there exist $x_n \in \overline{\Omega}, n = 1, \cdots, N$ and $x^*_n \in D^-_\beta f_n(x_n)$*

satisfying

$$(i)\ diam(x_1, \cdots, x_N) \max(1, \|x_1^*\|, \cdots, \|x_N^*\|) < \varepsilon;$$

$$(ii)\ \sum_{n=1}^{N} f_n(x_n) < \inf_{x \in \overline{\Omega}} \sum_{n=1}^{N} f_n(x) + \varepsilon;$$

$$(iii)\ \left\| \sum_{n=1}^{N} x_n^* \right\| < \varepsilon.$$

Conclusion

In Chapter 1, we have focused on the following:
1) We have given some remarks about the β-differentiable, the relationship between the β-differentiable when the borno β is implicit. We have also provided several remarks on common subdifferentials and their relations. In addition, we have pointed out certain cases in which the different functions have the same set of subdifferentials.
2) We have proved the addition rules of m sums of β-subdifferential.

Chapter 2

β-VISCOSITY SOLUTIONS OF HAMILTON-JACOBI EQUATIONS IN BANACH SPACES

Our main objective in this chapter is to prove the uniqueness of β-viscosity (which is weaker than Fréchet-viscosity) for Hamilton-Jacobi equations of the forms $u+H(x,Du)=0$ and $u+H(x,u,Du)=0$ on a set $\Omega\subset X$ the doubling of variables technique. Our results are established on a Banach space X with a β-smooth norm or a norm being equivalent to a β-smooth norm without using the Radon-Nikodym assumption. We also show the existence, uniqueness, and the stability of the solution. The results in this chapter are based on the paper [1] in the list of scientific publications related to this dissertation. In this dissertation, the solution existence of Dirichlet problem is proved under an additional assumption that there are equal subsolution and a supersolution on the boundary (compared with the existence result in [1]). In addition, we prove another result on the existence of a solution for Hamilton-Jacobi equations (Theorem 2.2.2).

2.1. The uniqueness of β-viscosity solutions

Let X be a real Banach space with a β-smooth norm $|\cdot|$, $\Omega\subset X$ an open subset. We study the existence, uniqueness and stability of β-viscosity solutions for the following HJEs

$$u+H(x,u,Du)=0 \text{ in } \Omega, \tag{2.1}$$

subject to the boundary condition (in the case $\Omega\neq X$)

$$u=\varphi \text{ on } \partial\Omega. \tag{2.2}$$

Here, $u:\Omega\to\mathbb{R}$ and $\varphi:\partial\Omega\to\mathbb{R}$ and $H:\overline{\Omega}\times\mathbb{R}\times X^*_\beta\to\mathbb{R}$ are merely continuous in general, where X^*_β is the dual space of the Banach space X, and equipped with topology τ_β (see Definition **??**).

2.1.1. β-viscosity solutions

Definition 2.1.1. *A function $u:\Omega\to\mathbb{R}$ is said to be* *(i)* *a* β*-viscosity subsolution of* (2.1) *if* u *is upper semicontinuous and for any* $x \in \Omega$, $x^* \in D^+_\beta u(x)$, $F(x, u(x), x^*) \leq 0$;

(ii) *a* β*-viscosity supersolution of (*(2.1)*) if* u *is lower semicontinuous and for any* $x \in \Omega$, $x^* \in D^-_\beta u(x)$, $F(x, u(x), x^*) \geq 0$;

(iii) *a* β*-viscosity solution of (*(2.1)*) if* u *is simultaneously a* β*-viscosity subsolution and a* β*-viscosity supersolution.*

For convenience, hereafter, we will use the phrases "β-viscosity solution of $H \leq 0$" and "β-viscosity subsolution of $H = 0$" interchangeably. Similarly for the phrases "β-viscosity solution of $H \geq 0$" and "β-viscosity supersolution of $H = 0$".

Definition 2.1.2. *A function* $u : \Omega \to \mathbb{R}$ *is said to be a* β*-viscosity subsolution (resp. supersolution, solution) of the problem* (2.1)-(2.2) *iff* u *is a* β*-viscosity subsolution (resp. supersolution, solution) of Equation* (2.1) *and* $u \leq \varphi$ *(resp.* $u \geq \varphi, u = \varphi$*) on* $\partial\Omega$.

Next, we make the following assumptions on the function H.

(H0) There exists a continuous function $w_R : X^*_\beta \to \mathbb{R}$ for each $R > 0$, satisfying

$$|H(x, r, p) - H(x, r, q)| \leq w_R(p - q)$$

whenever $x \in X$, $p, q \in X^*$ and $r \in \mathbb{R}$ satisfy $|x|, |q|, |p| \leq R$.

(H1) For each $(x, p) \in X \times X^*$, $r \mapsto H(x, r, p)$ is nondecreasing.

(H1)* For each $(x, p) \in X \times X^*$, $r \mapsto H(x, r, p)$ is Lipschitz continuous with constant $L < 1$.

(H2) There is a local modulus σ_H such that

$$H(x, r, p) - H(x, r, p + q) \leq \sigma_H(|q|, |p| + |q|)$$

for all $r \in \mathbb{R}$, $x \in \Omega$ and $p, q \in X^*$.

(H3) There is a modulus m_H such that

$$\begin{aligned} H(y, r, \lambda(\nabla_\beta|\cdot|^2)(x - y)) - &H(x, r, \lambda(\nabla_\beta|\cdot|^2)(x - y)) \\ &\leq m_H(\lambda|x - y|^2 + |x - y|) \end{aligned} \tag{2.3}$$

for all $x, y \in \Omega$ with $x \neq y$, $r \in \mathbb{R}$ and $\lambda \geq 0$.

2.1.2. Bounded solutions

Theorem 2.1.3. *Let X be a Banach space with an equivalent β-smooth norm. Suppose that $F(x,u,Du) = u + H(x,Du)$ with $H : X \times X^*_\beta \to \mathbb{R}$ satisfy the following assumption:*
(B) for any $x, y \in X$ and $x^*, y^* \in X^*_\beta$,

$$|H(x,x^*) - H(y,y^*)| \leq w(x-y, x^*-y^*) + K \max(\|x^*\|, \|y^*\|)\|x-y\|,$$

where K is a constant and $w : X \times X^*_\beta \to \mathbb{R}$ is continuous function with $w(0,0) = 0$.
Let u, v be two bounded functions such that u is upper semicontinuous and v is lower semicontinuous. If u is a β-viscosity subsolution and v is a β-viscosity supersolution of equations $F(x,u,Du) = 0$ then $u \leq v$.

Corollary 2.1.4. *Under the assumptions of Theorem 2.1.3, β-viscosity solutions continuous and bounded of equations $u + H(x,Du) = 0$ is unique.*

Theorem 2.1.5. *Let X be a Banach space with an equivalent β-smooth norm. $\Omega \subset X$ an open subset.*
Suppose $F(x,u,Du) = u + H(x,Du)$ with $H : X \times X^*_\beta \to \mathbb{R}$ satisfy the following assumption:
(C) for any $x, y \in X$ and $x^*, y^* \in X^*_\beta$,

$$|H(x,x^*) - H(y,y^*)| \leq w(x-y, x^*-y^*) + K \max(\|x^*\|, \|y^*\|)\|x-y\|,$$

wwhere K is a constant and $w : X \times X^*_\beta \to \mathbb{R}$ is continuous function with $w(0,0) = 0$.

Let u, v be two uniformly continuous bounded on $\overline{\Omega}$. If u is a β-viscosity subsolution and v is a β-viscosity supersolution of equations $F(x,u,Du) = 0$ and $u \leq v$ on $\partial\Omega$ then $u \leq v$ on $\overline{\Omega}$.

Corollary 2.1.6. *Under the assumptions of Theorem 2.1.5, u, v be two uniformly continuous bounded on $\overline{\Omega}$ such that $u = v$ on $\partial\Omega$. If u, v be two β-viscosity solution $F(x,u,Du) = 0$ then $u = v$ on $\overline{\Omega}$.*

2.1.3. Unbounded solutions

Based on the preparation in the preceding sections, now we present the main results on the uniqueness of the β-viscosity of (2.1).

Theorem 2.1.7. *Let X be a Banach space with a β-smooth norm, and Ω a open subset of X. Assume that the function H satisfies assumptions (H0)-(H3), $\widehat{H}$ satisfies (H0). Let $u, v \in C(\overline{\Omega})$ respectively be β-viscosity solutions of the problems*

$$u + H(x, u, Du) \leq 0 \;\; v \;\; v + \widehat{H}(x, v, Dv) \geq 0 \;\; on\ \Omega, \tag{2.4}$$

and assume that there is a modulus m such that

$$|u(x) - u(y)| + |v(x) - v(y)| \leq m(\|x - y\|) \;\; on\ \Omega. \tag{2.5}$$

Then, we have

$$\begin{aligned} & u(x) - v(x) \leq \sup_{\partial\Omega}(u - v)^+ + \sup_{\Omega\times\mathbb{R}\times X^*} (\widehat{H} - H)^+, \\ \Big(resp. \quad & u(x) - v(x) \leq \sup_{\partial\Omega}(u - v)^+ + \frac{1}{1 - L_H} \sup_{\Omega\times\mathbb{R}\times X^*} (\widehat{H} - H)^+\Big), \end{aligned} \tag{2.6}$$

for all $x \in \Omega$.

In particularly, when $\Omega = X$, we have estimate (2.6) *in which the term $\sup_{\partial\Omega}(u - v)^+$ on the right hand side is replaced by zero.*

Corollary 2.1.8 (Comparison and the uniqueness)**.** *Given X a Banach space, and equipped with a β-smooth norm. Let $\Omega \subset X$ be an open set with boundary $\partial\Omega \neq \emptyset$, φ a continuous function on $\partial\Omega$. Assume that the function H satisfies the assumptions (H0), (H1) (resp (H1)*), (H2), and (H3). If $u, v \in C(\overline{\Omega})$ respectively are β-viscosity subsolution and β-viscosity supersolution of Equation* (2.1) *satisfying* (2.5)*, then $u \leq v$ in Ω, provided that $u \leq v$ on $\partial\Omega$. Therefore, the problem* (2.1), (2.2) *has at most a solution in $C(\overline{\Omega})$.*

In the case Ω is the whole space X, the comparison and the uniqueness of the solution for Equation (2.1) *is an obvious consequence.*

2.2. The stability and uniqueness of β-viscosity solution

2.2.1. The stability

We proceed to study the stability of the β-viscosity solution. Using this stability in the same way as in [R. Deville, G. Godefroy, V. Zizler, (1993)], we obtain Proposition 2.2.1.

Theorem 2.2.1 (Stability). *Let X be a Banach space with a β-smooth norm, and Ω an open subset of X. Let $u_n \in C(\Omega)$ and $H_n \in C(\Omega \times \mathbb{R} \times X^*_\beta)$, $n = 1, 2, ...$ converge to u, H respectively as $n \to \infty$ in the following way:*

For every $x \in \Omega$ there is an $R > 0$ such that $u_n \to u$ uniformly on $B_R(x)$ as $n \to \infty$, and if $(x, r, p), (x_n, r_n, p_n) \in \Omega \times \mathbb{R} \times X^*_\beta$ for $n = 1, 2, ...$ and $(x_n, r_n, p_n) \to (x, r, p)$ as $n \to \infty$, then $H_n(x_n, r_n, p_n) \to H(x, r, p)$. If u_n is a β-viscosity supersolution (respectively, subsolution) of $H_n = 0$ in Ω, then u is a β-viscosity supersolution (respectively, subsolution) of $H = 0$ in Ω.

2.2.2. The existence

Theorem 2.2.2 (Existence). *Let X be a Banach space with a β-smooth norm, and Ω an open subset of X. Let $H : \Omega \times \mathbb{R} \times X^* \to \mathbb{R}$ satisfy (H0), (H1) (respectively (H1)*), (H2), (H3) and*

$$\liminf_{\|p\| \to \infty} (r + H(x, r, p)) > 0 \text{ uniformly for } (x, r) \in \Omega \times \mathbb{R}. \quad (2.7)$$

Then there exists a unique β-viscosity solution of the equations (2.1)

Theorem 2.2.3 (Existence of solutions for the Dirichlet problem). *Under the assumptions of Theorem 2.2.2 and suppose further that exists $u_0, v_0 \in C(\overline{\Omega})$ such that $u_0 = v_0 = \varphi$ on $\partial\Omega$; u_0, v_0 respectively a β-viscosity subsolution and β-viscosity supersolution of* (2.1) *then there exists a unique β-viscosity solution $u \in C(\overline{\Omega})$ of the problem* (2.1)-(2.2).

Conclude

In Chapter 2, we have focused on the following:
1. We have proved the uniqueness of β-viscosity solution for Hamilton-Jacobi equations.
2. We have investigated the stability of β-viscosity solutions for Hamilton-Jacobi equations.
3. We have shown the existence of β-viscosity solutions for Hamilton-Jacobi equations.

Chapter 3

APPLICATION OF THE β-VISCOSITY SOLUTIONS FOR OPTIMAL CONTROL PROBLEMS

In this chapter, we show that the value function of a certain infinite horizon optimal control problem is the unique β-viscosity solution of an associated Hamilton-Jacobi equation. Note that the boundedness of the solution is not needed in our proof. Moreover, we provide a necessary and sufficient condition for optimality in infinite dimensional spaces by using β-viscosity solution approach. The results in this chapter are based on the paper [2] in the List of scientific publications related to this dissertation.

3.1. The infinite horizon optimal control problems

3.1.1. Optimal control problems-dynamic programming principle Bellman with the value function smooth

Let X be a Banach space with a β-smooth norm and U be a metric space. Consider the following state equation:

$$\begin{cases} y'(s) = g(y(s), \alpha(s)), \quad s > 0, \\ y(0) = x, \ \alpha(s) \in U, \end{cases} \tag{3.1}$$

twhere $x \in X$ and $g : X \times U \to X$ is a given map with the control

$$\alpha(\cdot) \in \mathcal{U} := \{\alpha : [0, \infty) \to U \text{ measurable and } \alpha(t) \in U \text{ with } t \in [0, \infty) \text{ a.e.}\}.$$

We introduce the cost functional

$$J(x, \alpha) = \int_0^\infty e^{-\lambda s} f(y_x(s), \alpha(s)) ds, \tag{3.2}$$

where $\lambda > 0$ and $f : X \times U \to \mathbb{R}$. The optimal control problem $P(x)$ on X is to find $\overline{\alpha}(\cdot) \in \mathcal{U}$ such that

$$J(x, \overline{\alpha}(\cdot)) = \inf_{\alpha \in \mathcal{U}} J(x, \alpha).$$

We denote the value function of $P(x)$ by $V(x)$; that is,

$$V(x) = \inf_{\alpha \in \mathcal{U}} J(x, \alpha) = \inf_{\alpha \in \mathcal{U}} \left(\int_0^\infty e^{-\lambda s} f(y_x(s), \alpha(s)) ds \right).$$

Now, let us make several assumptions: The functions $g : X \times U \to X$ and $f : X \times U \to \mathbb{R}$ are continuous and satisfy one of the following set of conditions.

(B1) There exist constants $L_0, L, C, m > 0$, $K \in \beta$, with $0 \le m < \frac{\lambda}{L}$, $K \subset B(0, L)$ and a local modulus of continuity $\omega(\cdot, \cdot)$, such that for all $x, \overline{x} \in X$ and $u \in U$,

$$\begin{cases} |g(x,u) - g(\overline{x}, u)| \le L_0 \|x - \overline{x}\|, \quad g(x,u) \in K, \\ |f(x,u)| \le Ce^{m|x|}, \quad |f(x,u) - f(\overline{x}, u)| \le \omega(|x - \overline{x}|, |x| \vee |\overline{x}|), \end{cases}$$

where $|x| \vee |\overline{x}| = \max\{|x|, |\overline{x}|\}$.

(B2) There exist constants $L_0, L, C, m > 0$, $K \in \beta$, with $0 \le m < \frac{\lambda}{L_0}$, $K \subset B(0, L)$ and a local modulus of continuity $\omega(\cdot, \cdot)$, such that for all $x, \overline{x} \in X$ and $u \in U$,

$$\begin{cases} |g(x,u) - g(\overline{x}, u)| \le L_0 |x - \overline{x}|, \quad g(0,u) \in K, \\ |f(x,u)| \le C(1 + |x|)^m, \quad |f(x,u) - f(\overline{x}, u)| \le \omega(|x - \overline{x}|, |x| \vee |\overline{x}|). \end{cases}$$

3.1.2. Properties of the value function of the optimal control problem

Proposition 3.1.1 (X.J. Li, J.M. Yong, (1995), Proposition 6.1)**.** *Let one of (B1) or (B2) hold. Then, for any $x \in X$ and $u(\cdot) \in \mathcal{U}[0, \infty)$, the state equation* (3.1) *admits a unique trajectory $y_x(\cdot)$ and the cost functional* (3.2) *is well-defined. Moreover, we have the following:*

(a) *If (B1) holds, then V is locally uniformly continuous and for some constant $M > 0$,*

$$|V(x)| \le Me^{m|x|}, \quad x \in X.$$

(b) *If (B2) holds, then V is locally uniformly continuous and for some constant $M > 0$,*

$$|V(x)| \le M(1 + |x|)^m, \quad x \in X.$$

3.2. Application of the β-viscosity to the optimal control problem

We consider the optimal control problem (3.1)-(3.2). Define $H : X \times X_\beta^* \to \mathbb{R}$ by

$$H(x,p) = \sup_{\alpha \in \mathcal{U}} \{-\langle p, g(x,\alpha)\rangle - f(x,\alpha)\}.$$

Proposition 3.2.1. *(a) If (B1) holds, then*

$$\begin{cases} |H(x,p) - H(x,q)| \leq L|p-q|, \\ |H(x,p) - H(y,p)| \leq L_0|p||x-y| + \omega(|x-y|, |x| \vee |y|). \end{cases} \tag{3.3}$$

(b) If (B2) holds, then

$$\begin{cases} |H(x,p) - H(x,q)| \leq (L + L_0|x|)|p-q|, \\ |H(x,p) - H(y,p)| \leq L_0|p||x-y| + \omega(|x-y|, |x| \vee |y|). \end{cases} \tag{3.4}$$

Theorem 3.2.2. *Let X be a Banach space with a β-smooth norm. Let one of (B1)-(B2) holds. Then the value function V is a unique β-viscosity solution of*

$$\lambda V(x) + H(x, DV(x)) = 0. \tag{3.5}$$

Theorem 3.2.3. *For all $\alpha(\cdot) \in \mathcal{U}$, the following function is nondecreasing:*

$$s \mapsto \int_0^s e^{-\lambda t} f(y_x(t,\alpha), \alpha(t))dt + e^{-\lambda s} V(y_x(s,\alpha)), \quad s \in [0,\infty).$$

Moreover this function is constant if and only if the control $\alpha(\cdot)$ is optimal for the initial position x.

Another important finding of the article is the following. It gives a sufficient condition for a control to be optimal by relying on the concept of β-viscosity solutions of the Hamilton-Jacobi equations.

Proposition 3.2.4. *If V is locally Lipschitz and for a.e. s there exists $p \in D_\beta^\pm V(y_x(s))$ satisfy the inequality*

$$\lambda V(y_x(s)) - \langle p, y_x'(s)\rangle - f(y_x(s), \alpha(s)) \leq 0,$$

then $\alpha(\cdot)$ is the optimal control for x, where $D_\beta^\pm V(z) = D_\beta^+ V(z) \cup D_\beta^- V(z)$.

The following proposition provides an important result on feedback controls. Our approach is to employ β-viscosity sub-and-super differentials.

Proposition 3.2.5. *If V is locally Lipschitz; $\alpha(\cdot)$ is optimal for x, then*

$$\lambda V(y_x(s)) - \langle p, y_x'(s)\rangle - f(y_x(s), \alpha(s)) = 0$$

hold for all $p \in D_\beta^{\pm} V(y_x(s))$ for a.e. s.

From the above results we have the following theorem.

Theorem 3.2.6. *Assume V is locally Lipschitz and $D_\beta^{\pm} V(y_x(s)) \neq \emptyset$ for a.e. $s > 0$. Then the following statements are equivalent:*

(a) *$\alpha(\cdot)$ is optimal for x;*

(b) *for a.e. $s > 0$ and forall $p \in D_\beta^{\pm} V(y_x(s))$,*

$$\lambda V(y_x(s)) - \langle p, y_x'(s)\rangle - f(y, \alpha) = 0; \tag{3.6}$$

(c) *for a.e. $s > 0$ exists $p \in D_\beta^{\pm} V(y_x(s))$ such that (3.6) holds.*

Conclude

In this chapter, we have established the uniqueness of β-viscosity solutions for a class of HJEs. Compared to [J.M. Borwein, Q.J. Zhu, (1996)], our β-viscosity solutions are unbounded whereas those in [J.M. Borwein, Q.J. Zhu, (1996)] are bounded. We have also studied an optimal control problem with unbounded value functions, which can be regarded an extension of the results presented in [J.M. Borwein, Q.J. Zhu, (1996)]. Necessary and sufficient optimality conditions have been derived based on the β-viscosity solution approach. In this work, we have focused on infinite horizon optimal control problems. By similar approach and techniques presented in this paper, one can investigate finite-horizon, minimum time, and discounted minimum time control problems.

Chapter 4

HAMILTON-JACOBI EQUATIONS FOR OPTIMAL CONTROL ON JUNCTIONS WITH UNBOUNDED RUNNING COST FUNCTIONS

In this chapter, we study a class of optimal control problems on junctions. We show that the value function is a unique viscosity solution of an associated Hamilton-Jacobi equation. Moreover, properties of the value function are derived including the continuity, the growth, and upper bounds on the value function at the point O. We also establish a necessary and sufficient criterion of an optimal control for optimal control problems with infinite time horizon. The results of this chapter are based on the article [3] in the List of scientific publications related to the dissertation.

4.1. Optimal control problem on junctions

4.1.1. The junctions

We work with a model of junction in $\mathbb{R}^d$ with N semi-infinite straight edges, where N is a positive integer. For each $i = 1, \cdots, N$, we denote by e_i the standard unit vector in the ith direction and the edge $J_i = \mathbb{R}^+ e_i$. Then, the junction $\mathcal{G}$ is given by $\mathcal{G} = \bigcup_{i=1}^N J_i$.

4.1.2. The optimal control problem

One distinct feature of the problem under consideration is that on different edges of the junction $\mathcal{G}$, one observes different dynamics and running costs. For this, we denote by A_i the set of control on J_i, while f_i and l_i are the control mapping and the running cost on J_i, respectively. To proceed, we introduce assumptions that will be used in this paper.

(H0) Let A be a metric space. Assume that $A_1, A_2, \cdots A_N$ are nonempty compact subsets of A and the sets A_i are disjoint.

(H1) For each $i = 1, \cdots, N$, the function $f_i : J_i \times A_i \to \mathbb{R}$ is continuous and bounded by a positive constant M. Moreover, there exists $L > 0$ such that $|f_i(x, a) - f_i(y, a)| \le L|x - y|$ for all $x, y \in J_i, a \in A_i$.

We denote by $F_i(x)$ the set $\{f_i(x,a)e_i : a \in A_i\}$.
(H2) For each $i = 1, \cdots, N$, the function $\ell_i : J_i \times A_i \to \mathbb{R}$ is continuous. Moreover, there are constants $C, m \geq 0$, with $0 \leq m < \frac{\lambda}{M}$, where $\lambda > 0$ is a constant and a local modulus of continuity $\omega(\cdot,\cdot)$ such that

$$|\ell_i(x,a) - \ell_i(y,a)| \leq \omega(|x-y|, |x| \vee |y|) \text{ for all } x, y \in J_i, a \in A_i,$$

$$|\ell_i(x,a)| \leq Ce^{m|x|} \text{ for all } x \in J_i, a \in A_i,$$

(H2)* For each $i = 1, \cdots, N$, the function $\ell_i : J_i \times A_i \to \mathbb{R}$ is continuous and there are constants $C, m \geq 0$ and a local modulus of continuity $\omega(\cdot,\cdot)$ such that

$$|\ell_i(x,a) - \ell_i(y,a)| \leq \omega(|x-y|, |x| \vee |y|) \text{ for all } x, y \in J_i, a \in A_i,$$

$$|\ell_i(x,a)| \leq C(1+|x|)^m \text{ for all } x \in J_i, a \in A_i.$$

where the number M in (H2) and (H2)* is given in (H1),
(H3) For each $i = 1, \cdots, N$, the following set

$$FL_i(x) = \{(f_i(x,a)e_i, \ell_i(x,a)) : a \in A_i\}$$

is non-empty, closed and convex.
(H4) There exists a real number $\delta > 0$ such that

$$[-\delta e_i, \delta e_i] \subset F_i(O) = \{f_i(O,a)e_i : a \in A_i\}.$$

We define

$$\mathcal{M} = \{(x,a) : x \in \mathcal{G}, a \in A_i \text{ if } x \in J_i \backslash \{O\}, \text{ and } a \in \cup_{i=1}^N A_i \text{ if } x = O\}.$$

Then $\mathcal{M}$ is closed and a function $f(\cdot,\cdot)$ defined on $\mathcal{M}$ is given by

$$\text{for any } (x,a) \in \mathcal{M}, \quad f(x,a) = \begin{cases} f_i(x,a)e_i & \text{if } x \in J_i \backslash \{O\}, \\ f_i(O,a)e_i & \text{if } x = O \text{ and } a \in A_i. \end{cases}$$

Since the functions $f_i : J_i \times A_i \to \mathbb{R}$ are continuous and the sets A_i are disjoint, then f is continuous on $\mathcal{M}$. Let $\widetilde{F}(x)$ be defined by

$$\widetilde{F}(x) = \begin{cases} F_i(x) & \text{if } x \in J_i \backslash \{O\} \\ \cup_{i=1}^N F_i(O) & \text{if } x = O. \end{cases}$$

For each $x \in \mathcal{G}$, the collection of admissible trajectories starting from x is given by

$$Y_x = \left\{ y_x \in Lip(\mathbb{R}^+;\mathcal{G}) : \left| \begin{array}{l} \dot{y}_x(t) \in \widetilde{F}(y_x(t)) \text{ for a.e. } t > 0 \\ y_x(0) = x \end{array} \right. \right\}.$$

The set of admissible controlled trajectories starting from x is given by

$$\begin{aligned} \mathcal{T}_x = \big\{ (y_x, \alpha) \in L^\infty_{loc}(\mathbb{R}^+;\mathcal{M}) : & y_x \in Lip(\mathbb{R}^+;\mathcal{G}) \text{ and} \\ & y_x(t) = x + \int_0^t f(y_x(s), \alpha(t)) ds \big\} \end{aligned}$$

Let $\lambda > 0$ be a real number and the cost function ℓ be defined on $\mathcal{M}$ by $\forall (x,a) \in \mathcal{M}, \quad \ell(x,a) = \begin{cases} \ell_i(x,a) & \text{if } x \in J_i \backslash \{O\}, \\ \ell_i(O,a) & \text{if } x = O \text{ and } a \in A_i. \end{cases}$ Then the cost functional associated with the trajectory $(y_x, \alpha) \in \mathcal{T}_x$ is

$$v(x) = \inf_{(y_x,\alpha) \in \mathcal{T}_x} J(x; (y_x, \alpha)). \tag{4.1}$$

4.1.3. Some properties of the value function at the vertex

Lemma 4.1.1. *Under assumption (H0), (H1), (H2) or* $(H2)^*$*, (H3), (H4), there exists* $\varepsilon > 0$ *such that* $v|_{J_i}$ *is Lipschitz continuous in* $J_i \cap B(O, \varepsilon)$.

Lemma 4.1.2. *Under assumption (H0), (H1), (H2) or* $(H2)^*$*, (H3), (H4), the value function* v *satisfies* $v(O) \leq -\frac{H_O^T}{\lambda}$.

4.2. The HJe and viscosity solutions

4.2.1. Test-functions

To proceed, we recall the definition of the admissible test-functions.

Definition 4.2.1. *A function* $\varphi : \mathcal{G} \to \mathbb{R}$ *is called an admissible test-function if it satisfies the following two conditions.*

a) φ *is continuous in* $\mathcal{G}$ *and continuously differentiable in* $\mathcal{G} \backslash \{O\}$,

b) for any $j = 1, \cdots, N$, $\varphi|_{J_j}$ *is continuously differentiable in* J_j.

Let $\mathcal{R}(\mathcal{G})$ be the set of admissible test-functions. For any $\varphi \in \mathcal{R}(\mathcal{G})$ and $\zeta \in \mathbb{R}$, let $D\varphi(x, \zeta e_i)$ be defined by

$$D\varphi(x, \zeta e_i) = \begin{cases} \zeta \frac{d\varphi}{dx_i}(x) & \text{if } x \in J_i \backslash \{O\} \\ \zeta \lim_{h \to 0^+} \frac{d\varphi}{dx_i}(he_i) & \text{if } x = O. \end{cases}$$

4.2.2. Vector fields

For $i = 1, \cdots, N$, we denote by $F_i^+(O)$ and $FL_i^+(O)$ the sets

$$F_i^+(O) = F_i(O) \cap \mathbb{R}^+ e_i, \quad FL_i^+(O) = FL_i(O) \cap (\mathbb{R}^+ e_i \times \mathbb{R}),$$

which are non empty based on assumption (H3). Note that $0 \in \cap_{i=1}^N F_i(O)$. From assumption (H2), these sets are compact and convex. For $x \in \mathcal{G}$, the sets $F(x)$ and $FL(x)$ are defined by

$$F(x) = \begin{cases} F_i(x) & \text{if } x \text{ belongs to the edge } J_i \backslash \{O\} \\ \cup_{i=1}^N F_i^+(O) & \text{if } x = O, \end{cases}$$

$$FL(x) = \begin{cases} FL_i(x) & \text{if } x \text{ belongs to the edge } J_i \backslash \{O\} \\ \cup_{i=1}^N FL_i^+(O) & \text{if } x = O. \end{cases}$$

4.2.3. Definition of viscosity solutions

We now introduce the definition of a viscosity solution of

$$\lambda u(x) + \sup_{(\zeta,\xi) \in FL(x)} \{-Du(x, \zeta) - \xi\} = 0 \quad \text{in } \mathcal{G}. \tag{4.2}$$

Definition 4.2.2. *•A function $u : \mathcal{G} \to \mathbb{R}$ is said to be a viscosity subsolution of (4.2) in $\mathcal{G}$ if u is upper semi-continuous and for any $x \in \mathcal{G}$, $\varphi \in \mathcal{R}(\mathcal{G})$ such that $u - \varphi$ has a local maximum point at x, one has*

$$\lambda u(x) + \sup_{(\zeta,\xi) \in FL(x)} \{-D\varphi(x, \zeta) - \xi\} \leq 0. \tag{4.3}$$

•A function $u : \mathcal{G} \to \mathbb{R}$ is said to be a viscosity supersolution of (4.2) in $\mathcal{G}$ if it is lower semi-continuous and for any $x \in \mathcal{G}$, $\varphi \in \mathcal{R}(\mathcal{G})$ such that $u - \varphi$ has a local minimum point at x, one has

$$\lambda u(x) + \sup_{(\zeta,\xi) \in FL(x)} \{-D\varphi(x, \zeta) - \xi\} \geq 0. \tag{4.4}$$

• A continuous function $u : \mathcal{G} \to \mathbb{R}$ is a viscosity solution of (4.2) in $\mathcal{G}$ if it is both a viscosity subsolution and a viscosity supersolution of (4.2) in $\mathcal{G}$.

4.2.4. Hamilton function

We define the Hamilton $H_i : J_i \times \mathbb{R} \to \mathbb{R}$ by

$$H_i(x,p) = \max_{a \in A_i}\{-pf_i(x,a) - \ell_i(x,a)\},$$

and the Hamilton $H_O : \mathbb{R}^N \to \mathbb{R}$ by

$$H_O(p_1,\cdots,p_N) = \max_{i=1,\cdots,N} \max_{a \in A_i \text{ s.t. } f_i(O,a)\geq 0}\{-p_i f_i(O,a) - \ell_i(O,a)\}.$$

Definition 4.2.3. • *A function $u : \mathcal{G} \to \mathbb{R}$ is said to be a viscosity subsolution of* (4.2) *in $\mathcal{G}$ if it is upper semi-continuous and for any $x \in \mathcal{G}$, $\varphi \in \mathcal{R}(\mathcal{G})$ such that $u - \varphi$ has a local maximum point at x, one has*

$$\begin{aligned} &\lambda u(x) + H_i\left(x, \frac{d\varphi}{dx_i}(x)\right) \leq 0 \quad \text{if } x \in J_i \setminus\{O\}, \\ &\lambda u(O) + H_O\left(\frac{d\varphi}{dx_1}(O),\cdots,\frac{d\varphi}{dx_N}(O)\right) \leq 0. \end{aligned} \tag{4.5}$$

• *A function $u : \mathcal{G} \to \mathbb{R}$ is said to be lower semi-continuous if it is viscosity supersolution of* (4.2) *in $\mathcal{G}$ and for any $x \in \mathcal{G}$, $\varphi \in \mathcal{R}(\mathcal{G})$ such that $u - \varphi$ has a local minimum point at x, one has*

$$\begin{aligned} &\lambda u(x) + H_i\left(x, \frac{d\varphi}{dx_i}(x)\right) \geq 0 \quad \text{if } x \in J_i \setminus\{O\} \\ &\lambda u(O) + H_O\left(\frac{d\varphi}{dx_1}(O),\cdots,\frac{d\varphi}{dx_N}(O)\right) \geq 0. \end{aligned} \tag{4.6}$$

Theorem 4.2.4. *Assuming (H0), (H1), (H2) (or (H2)*) and (H3), the value function v defined in* (4.1) *is a viscosity solution of* (4.2) *in $\mathcal{G}$.*

4.3. Comparison Principle and Uniqueness

Theorem 4.3.1. *(a) Assume (H0), (H1), (H2) and (H3). Let $u, v : \mathcal{G} \to \mathbb{R}$ satisfy $|u(x)| \leq Ke^{m|x|}$, $|v(x)| \leq Ke^{m|x|}$ for some constant $K > 0$, $x \in \mathcal{G}$, with $0 \leq m < \frac{\lambda}{M}$, and u, v continuous on $\mathcal{G}$. Moreover, there exists $r_i > 0$ such that $u|_{J_i}, v|_{J_i}$ is Lipschitz continuous in $J_i \cap B(O, r_i)$. Suppose that u is a viscosity subsolution*

and v is a viscosity supersolution of (4.2) *in $\mathcal{G}$. Then $u \le v$.*
(b) Assume (H0), (H1), (H2)* and (H3). Let $u, v : \mathcal{G} \to \mathbb{R}$ satisfy

$$|u(x)| \le K(1+|x|)^m, \ |v(x)| \le K(1+|x|)^m$$

for some constant $K > 0$, $x \in \mathcal{G}$, with $0 \le m$, and u, v continuous on $\mathcal{G}$. Moreover, there exists $r_i > 0$ such that $u|_{J_i}, v|_{J_i}$ is Lipschitz continuous in $J_i \cap B(O, r_i)$. Suppose that u is a viscosity subsolution and v is a viscosity supersolution of (4.2) *in $\mathcal{G}$. Then $u \le v$.*

4.4. Applications of viscosity solutions

Theorem 4.4.1. *For all $x \in \mathcal{G}$, and $(y_x, \alpha) \in \mathcal{T}_x$, the following function is nondecreasing:*

$$s \mapsto \int_0^s e^{-\lambda t} f(y_x(t), \alpha(t))dt + e^{-\lambda s} v(y_x(s)), \quad s \in [0, \infty).$$

Moreover this function is constant if and only if the control $\alpha(.)$ is optimal for the initial position x.

Theorem 4.4.2. *For every $x \in \mathcal{G}$, if $\alpha(\cdot)$ is a control such that for $(y_x, \alpha) \in \mathcal{T}_x$, and the value function v is Lipschitz continuous and satisfies*

$$\liminf_{t \to 0^+} \frac{v(y_x(s) + tf(y_x(s), \alpha(s))) - v(y_x(s))}{t} + \ell(y_x(s), \alpha(s)) \le \lambda v(y_x(s)) \tag{4.7}$$

for almost all s, then $\alpha(\cdot)$ is optimal for the initial position x.

Theorem 4.4.3. *Suppose that the value function v is locally Lipschitz. Then $\alpha(.)$ is optimal for the initial position x iff*

$$\lim_{t \to 0} \frac{v(y_x(s) + tf(y_x(s), \alpha(s))) - v(y_x(s))}{t} + \ell(y_x(s), \alpha(s)) = \lambda v(y_x(s)) \tag{4.8}$$

for a.e. s.

Conclusion

In Chapter 4, we have focused on a class of optimal control problems on junctions. Compared to recent works, in our formulation the running costs belong to a broader class of functions. As a result, the value function might be unbounded. We have also established a necessary and sufficient criterion of an optimal control for optimal control problems with infinite time horizon.

CONCLUSION

This dissertation is devoted to applications of subdifferential for viscosity solutions to Hamilton-Jacobi equations in Banach spaces. In particular, the dissertation focuses on the following: (1) β-viscosity subdifferential, properties of β-viscosity subdifferential, the smooth variational principle; (2) the uniqueness of β-viscosity solution for Hamilton-Jacobi equations in Banach spaces of the form $u+H(x,Du) = 0$ and $u+H(x,u,Du) = 0$; the existence and stability of β-viscosity solution; (3) β-viscosity solution for optimal control problems in Banach spaces, and feedback controls for infinite horizon optimal control problems; (4) viscosity solutions for optimal control problems on junctions, the necessary and sufficient conditions for optimality. The results of this dissertation can be summarized as follows.

1. We have proved several results on the smooth variational principle for upper semicontinuous and bounded functions on a Banach space X satisfying assumption (H^*_β) and with a β-smooth norm on the space.
2. We have proved the uniqueness of β-viscosity solution of Hamilton-Jacobi equations in the class of continuous and bounded functions for Hamilton-Jacobi equations of the form $u+H(x,Du) = 0$, the uniqueness of the solution in the class of uniformly continuous and unbounded for first order partial differential equations of general form $u + H(x,u,Du) = 0$. We also established the existence and stability of β-viscosity solution for Hamilton-Jacobi equations of the form $u + H(x,u,Du) = 0$.
3. We have shown that the value function of a certain infinite horizon optimal control problem is the unique β-viscosity solution of an associated Hamilton-Jacobi equation. In addition, we have established a necessary and sufficient condition for optimality.
4. Studying the viscosity solution on junctions, we have proved that the value function is continuous and bounded above at O. We have also provided a necessary and sufficient condition for optimality of a certain infinite horizon optimal control problem.

LIST OF SCIENTIFIC PUBLICATIONS RELATED TO THE DISERTATION

[1] T.V. Bang, P.T. Tien, (2018), On the existence, uniqueness, and stability of β-viscosity solutions to a class of Hamilton-Jacobi equations in Banach spaces, *Acta Math. Vietnam* DOI: 10.1007/s40306-018-0287-7

[2] P.T. Tien, T.V. Bang, (2019), Uniqueness of β-viscosity solutions of Hamilton-Jacobi equations and applications to a class of optimal control problems, *Differ. Equ. Dyn. Syst.* DOI: 10.1007/s12591-019-00479-7

[3] P.T. Tien, T.V. Bang, (2019), Hamilton-Jacobi equations for optimal control on junctions with unbounded running cost functions, *Appl. Anal.* DOI: 10.1080/00036811.2019.1643012.