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STABILITY AND ROBUST STABILITY OF LINEAR DYNAMIC EQUATIONS ON TIME SCALES

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SUMMARY

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INTRODUCTION

In 1988, the analysis on time scales was introduced by Stefan Hilger in his Ph.D. dissertation in order to build bridges between continuous and discrete systems and unify two these ones. One of the most important problems is to consider the stability of dynamic equations. There have been a lot of works on the theory of time scales published over the years. The dissertation's content has mentioned three problems: Lyapunov exponent, Bohl exponent, and stability radius of dynamic equations.

Bohl exponent, Lyapunov exponent investigate the asymptotic behavior of solutions of differential equations. Lyapunov exponent is introduced by A. M. Lyapunov (1857-1918) in his Ph.D. dissertation in 1892, Bohl exponent, by P. Bohl (1865-1921) in 1913 in his article[1]. Both of them describe the exponential growth of solutions of dynamic equations on time scales, $\dot{x} = A(t)x$.

The first Lyapunov method (or Lyapunov exponent method) was a quite classical and basic concept for studying differential and difference equations, see Li, Yang, and Zhang (2014), Martynyuk (2013, 2016), and it is a useful tool to study the stability of linear systems. But so far, there has been no work dealing with the concept of Lyapunov exponents and the stability for functions defined on time scales. The main reason is that the traditional approach to Lyapunov exponents via logarithm functions is no longer valid. Because, there is no reasonable definition for logarithm functions on time scales, which one regards as the inverse of the exponential function $e_{p(t)}(t,s)$.

We study the first Lyapunov method for dynamic equations on time scales with a suitable approach, instead of considering the limit $\limsup_{t\to\infty} \frac{1}{t} \ln |f(t)|$, we will use the oscillation of the ratio $\frac{|f(t)|}{e_\alpha(t,t_0)}$ as $t \to \infty$ in the parameter α to define the Lyapunov exponent of a function f on time scales, and use it to investigate the stability of dynamic equations

$$x^\Delta = A(t)x$$

on time scales. We obtain some main results, such as the definition of Lyapunov exponent $\kappa_L[f(\cdot)]$ of the function $f(\cdot)$, the sufficient and necessary condition for the existence of $\kappa_L[f(\cdot)]$, the sufficient condition on the boundedness of Lyapunov exponent $\kappa_L[x(\cdot)]$, where $x(\cdot)$ is a nontrivial solution of the equation $x^\Delta = A(t)x$, the sufficient conditions on the stability of the equation $x^\Delta = A(t)x$, where matrix $A(\cdot)$ is bounded or is a constant matrix, and specially, the spectrum condition for the exponential stability of this equation.

The Bohl exponent has been successfully used to characterize exponential stability

[1] Bohl P. (1913), Uber Differentialungleichungen, J.F.d. Reine Und Angew. Math., **144**, 284–133.

and to derive robustness results for ordinary differential equations (ODEs), see, e.g. Daleckii and Krein (1974), Hinrichsen et al. (1989). In Chyan et al. (2008), the authors generalized several results of the ODE concerning the Bohl exponent to linear differential-algebraic equations (DAEs) with index-1, $E(t)\dot{x} = A(t)x + f(t)$, where $E(\cdot)$ is supposed to be singular. In 2009, Linh and Mehrmann investigated Bohl spectral interval and Bohl exponent of particular solution and fundamental solution matrices of DAEs. However, the Bohl exponent of linear systems does not lie in both articles' focus, Chyan et al. (2008), Linh and Mehrmann (2009). In Berger's article (2012), the author developed the theory of Bohl exponents for linear time-varying differential-algebraic equations. The results of this paper are the generalizations of ODE results in Daleckii and Krein (1974), Hinrichsen et. al. (1989) and the others to DAEs. Recently, in 2016, Du et al. introduced the concept of Bohl exponents and characterized the relationship between the exponential stability and the Bohl exponent of linear singular systems of difference equations with variable coefficients.

The dissertation will introduce the Bohl exponent of implicit dynamic equations (IDEs) on time scale

$$E_\sigma(t)x^\Delta = A(t)x,$$

and characterize the relation between the exponential stability and the Bohl exponent. Some results are obtained, such as the solution formula of linear time-varying IDEs $E_\sigma(t)x^\Delta = A(t)x + f(t)$; the robust stability of IDEs subject to Lipschitz perturbations in Theorem 3.10, and the Bohl-Perron type stability theorem for these equations in Theorem 3.14; the concept of Bohl exponent and the relationship among the exponential stability, the Bohl exponent of equations $E_\sigma(t)x^\Delta = A(t)x$ and solutions of the respective Cauchy problem is derived, Theorem 3.23; the robustness of Bohl exponent when equations are subject to perturbations acting on the coefficients in Theorems 3.26 and 3.27.

The rest problem studied in the dissertation is the robust stability of IDEs on time scales. We know that, the stability radius of differential-algebraic or implicit difference equations is a subject that has attracted the attention of researchers. There have been many published works. However, results for the stability radius of time-varying systems are few. The concept of stability radius of linear time-varying systems is introduced in Hinrichsen et al. (1992),

$$r_{\mathbb{C}}(I, A; B, C) = \inf \left\{ \begin{array}{c} \|\Sigma\|_{L_\infty}, \Sigma \in \mathrm{PC_b}(\mathbb{R}^+, \mathbb{C}^{m\times q}) \\ \text{and } (I, A; B, C) \text{ is not exponential stable} \end{array} \right\},$$

and the first stability radius formula is derived in Jacob's article (1998),

$$r_{\mathbb{K}}(I, A; B, C) = \sup_{t_0 \geq 0} \|\mathbb{L}_{t_0}\|^{-1}.$$

In 2006, Du and Linh investigated the stability radius of linear time-varying DAEs having index-1 and obtained the stability radius formula,

$$r_{\mathbb{K}}(E, A; B, C) = \min \left\{ \sup_{t_0 \geq 0} \|\mathbb{L}_{t_0}\|^{-1}, \|\widetilde{\mathbb{L}}_0\|^{-1} \right\}.$$

In 2009, Rodjanadid et al. studied and derived the stability radius formula of linear time-varying implicit difference equation with index-1,

$$r_{\mathbb{K}}(E,A;B,C) = \min\left\{\sup_{n_0\geq 0}\|\mathbb{L}_{n_0}\|^{-1}, \|\widetilde{\mathbb{L}}_0\|^{-1}\right\}.$$

In 2014, Berger derived some lower bounds for the stability radii of time-varying DAEs of index-1 under unstructured perturbations acting on the coefficient of derivative,

$$r(E,A) \geq \begin{cases} \frac{\min\{l(E,A),\|QG^{-1}\|_\infty^{-1}\}}{\kappa_1+\kappa_2\min\{l(E,A),\|QG^{-1}\|_\infty^{-1}\}} & \text{if } Q\neq 0,\\ \frac{l(E,A)}{\kappa_1+\kappa_2 l(E,A)}, & \text{if } Q=0 \text{ and } l(E,A)<\infty,\\ \frac{1}{\kappa_2}, & \text{if } Q=0 \text{ and } l(E,A)=\infty. \end{cases}$$

The dissertation will investigate the stability, robust stability of linear time-varying IDEs on time scales $E_\sigma(t)x^\Delta(t) = A(t)x(t) + f(t)$, the corresponding homogeneous form $E_\sigma(t)x^\Delta(t) = A(t)x(t)$. We have investigated generally the robust stability for linear time-varying IDEs on time scales, and have also obtained some derived results, such as the formula of structured stability radius of IDEs with respect to dynamic perturbations, Theorem 4.9, and a lower bound, Corollary 4.13; the lower bounds for the stability radius involving structured perturbations acting on both sides, Theorem 4.20, and Corollary 4.22. Many previous results for the stability radius of time-varying differential, difference equations, differential-algebraic equations and implicit difference equations are also extended, Remarks 4.10, 4.11, 4.14, and 4.15.

The dissertation was completed at Hanoi Pedagogical University 2, Course 2015 - 2019 and presented at the seminar of the Faculty of Mathematics, HPU2. The results of dissertation were reported at

1. Vietnam - Korea Joint Workshop on Dynamical Systems and Related Topics (Vietnam Institute for Advanced Study in Mathematics, Hanoi, Vietnam, March 02-05, 2016);

2. the 2nd Pan-Pacific International Conference on Topology and Applications (Pusan National University, Busan, Korea, November 13-17, 2017);

3. the 9th Vietnam Mathematical Congress (Vietnamese Mathematical Society, Nhatrang, Vietnam, August 14-18, 2018); and

4. International Conference Differential Equations and Dynamical Systems (Hanoi Pedagogical University 2 and Institue of Mathematics - Vietnam Academy of Science and Technology, Vinhphuc, September 05-07, 2019).

CHAPTER 1

PRELIMINARIES

In this chapter, we introduce some basic concepts about the theory of analysis on time scales to study the stability and robust stability of dynamic equations. In 1988, the theory of analysis on time scales was introduced by Stefan Hilger in his Ph.D. dissertation in order to unify and extend continuous and discrete calculus. The content in Chapter 1 is referenced from Bohner M. and Peterson A. (2001), Bohner & Peterson (2003) and the material therein.

1.1 Time Scale and Calculations

1.1.1 Definition and Example

The *time scale* denoted by $\mathbb{T}$ is an arbitrary, nonempty, closed subset of the set of real numbers $\mathbb{R}$. We assume throughout that time scale $\mathbb{T}$ has a topology that inherited from the set of real numbers with the standard topology.

Definition 1.2 (Bohner & Peterson (2001), page 1). Let $\mathbb{T}$ be a time scale. For all $t \in \mathbb{T}$,

i) the *forward operator* $\sigma : \mathbb{T} \to \mathbb{T}$ by $\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}$,

ii) the *backward operator* $\varrho : \mathbb{T} \to \mathbb{T}$ by $\varrho(t) := \sup\{s \in \mathbb{T} : s < t\}$, and

iii) the *graininess* function $\mu : \mathbb{T} \to [0, \infty)$ by $\mu(t) = \sigma(t) - t$.

We define the so-called set $\mathbb{T}^\kappa$ as follows: $\mathbb{T}^\kappa = \begin{cases} \mathbb{T} \backslash (\varrho(\sup \mathbb{T}), \sup \mathbb{T}] & \text{if } \sup \mathbb{T} < \infty, \\ \mathbb{T} & \text{if } \sup \mathbb{T} = \infty. \end{cases}$

Definition 1.4 (Bohner & Peterson (2001), page 2). A point $t \in \mathbb{T}$ is said to be *left-dense* if $t > \inf \mathbb{T}$ and $\varrho(t) = t$; *right-dense* if $t < \sup \mathbb{T}$ and $\sigma(t) = t$, and *dense* if t is simultaneuosly right-dense and left-dense; *left-scattered* if $\varrho(t) < t$; *right-scattered* if $\sigma(t) > t$, and *isolated* if t is simultaneuosly right-scattered and left-scattered.

If $f : \mathbb{T} \to \mathbb{R}$ is a function, then $f_\sigma : \mathbb{T} \to \mathbb{R}$ is a function defined by $f_\sigma(t) := f(\sigma(t))$ for all $t \in \mathbb{T}$, i.e., $f_\sigma = f \circ \sigma$. Fix $t_0 \in \mathbb{T}$ and set $\mathbb{T}_{t_0} := [t_0, \infty) \cap \mathbb{T}$.

1.1.2 Differentiation

Definition 1.7 (Bohner & Peterson (2001), page 5)**.** A function $f : \mathbb{T} \to \mathbb{R}$ is called *delta differentiable* at t if there exists a function $f^{\Delta}(t)$ such that for all $\varepsilon > 0$,

$$|f(\sigma(t)) - f(s) - f^{\Delta}(t)(\sigma(t) - s)| \leq \varepsilon|\sigma(t) - s|,$$

for all $s \in U = (t - \delta, t + \delta) \cap \mathbb{T}$ and for some $\delta > 0$. The function $f^{\Delta}(t)$ is called the *delta* (or *Hilger*) *derivative* of f at the point t. When f is also called *delta* (or *Hilger*) *differentiable* on $\mathbb{T}^{\kappa}$. We use words *derivative, differentiable* to replace words *delta derivative, delta differentiable* if it is not confused.

1.1.3 Integration

Definition 1.16 (Bohner & Peterson (2001), page 22)**.** A function $f : \mathbb{T} \to \mathbb{R}$ is called *regulated* provided its right-sided limits exist (finite) at all right-dense points in $\mathbb{T}$ and its left-sided limits exist (finite) at all left-dense points in $\mathbb{T}$; *rd-continuous* provided it is continuous at right-dense points in $\mathbb{T}$ and its left-sided limits exist (finite) at left-dense points in $\mathbb{T}$.

The set of *rd*-continuous functions $f : \mathbb{T} \to \mathbb{R}$ will be denoted by $\mathrm{C_{rd}} = \mathrm{C_{rd}}(\mathbb{T}) = \mathrm{C_{rd}}(\mathbb{T}, \mathbb{R})$. The set of functions $f : \mathbb{T} \to \mathbb{R}$ that are differentiable and whose derivative is *rd*-continuous will be denoted by $\mathrm{C^1_{rd}} = \mathrm{C^1_{rd}}(\mathbb{T}) = \mathrm{C^1_{rd}}(\mathbb{T}, \mathbb{R})$. The set of *rd*-continuous functions defined on the interval J and valued in X is denoted by $\mathrm{C_{rd}(J, X)}$.

Definition 1.17 (Bohner & Peterson (2001), page 22)**.** A continuous function $f : \mathbb{T} \to \mathbb{R}$ is called *pre-differentiable* with (region of differentiation) D, provided $D \subset \mathbb{T}^{\kappa}$, $\mathbb{T}^{\kappa} \backslash D$ is countable and contains no right-scattered element of $\mathbb{T}$, and f is differentiable at each $t \in D$.

Definition 1.20 (Guseinov (2003))**.** Assume $f : \mathbb{T} \to \mathbb{R}$ is a regulated function.

i) The function F is called a *pre-antiderivative* of f if $F^{\Delta}(t) = f(t)$, $\forall\, t \in D$.

ii) The indefinite integral of f is defined by $\int f(t)\Delta t = F(t) + C$, where C is an arbitrary constant and F is a pre-antiderivative of f.

iii) The *Cauchy integral* of f is defined by $\int_a^b f(t)\Delta t = F(b) - F(a), \forall a, b \in \mathbb{T}$, where F is a pre-antiderivative of the function f.

iv) A function $F : \mathbb{T} \to \mathbb{R}$ is called an *antiderivative* of $f : \mathbb{T} \to \mathbb{R}$ provided $F^{\Delta}(t) = f(t)$ holds for all $t \in \mathbb{T}^{\kappa}$.

Theorem 1.25 (Bohner & Peterson (2003), page 46)**.** *Let $a \in \mathbb{T}^{\kappa}, b \in \mathbb{T}$ and suppose $f : \mathbb{T} \times \mathbb{T}^{\kappa} \to \mathbb{R}$ is continuous at (t, t), where $t \in \mathbb{T}^{\kappa}, t > a$. Also assume that $f^{\Delta}(t, \cdot)$*

is rd-continuous on the interval $[a, \sigma(t)]$. *Also suppose that* $f(\cdot, \tau)$ *is delta differentiable for each* $\tau \in [a, \sigma(t)]$. *Suppose that for every* $\varepsilon > 0$, *there exists a neighbourhood* U *of* t *such that* $|f(\sigma(t), \tau) - f(s, \tau) - f^{\Delta}(t, \tau)(\sigma(t) - s)| \leq \varepsilon|\sigma(t) - s|$, *for all* $s \in U$, *where* f^{Δ} *denotes the derivative of* f *with respect to the first variable. Then*

$$g(t) := \int_a^t f(t, \tau)\Delta\tau \;\; \textit{implies} \;\; g^{\Delta}(t) = f(\sigma(t), t) + \int_a^t f^{\Delta}(t, \tau)\Delta\tau.$$

1.1.4 Regressivity

Definition 1.26 (Bohner & Peterson (2003), page 10)**.** A function $p : \mathbb{T} \to \mathbb{R}$ is called *regressive*, if $1 + \mu(t)p(t) \neq 0$, for all $t \in \mathbb{T}^{\kappa}$; *positively regressive*, if $1 + \mu(t)p(t) > 0$, for all $t \in \mathbb{T}^{\kappa}$; and *uniformaly regressive*, if there exists a number $\delta > 0$ such that $|1 + \mu(t)p(t)| \geq \delta$, for all $t \in \mathbb{T}^{\kappa}$.

Denote $\mathcal{R} = \mathcal{R}(\mathbb{T}, \mathbb{R})$ (resp. $\mathcal{R}^{+} = \mathcal{R}^{+}(\mathbb{T}, \mathbb{R})$) the set of regressive (resp., positively regressive) functions on time scale $\mathbb{T}$.

1.2 Exponential Function

Definition 1.33 (Bohner & Peterson (2001), page 59)**.** If $p(\cdot) \in \mathcal{R}$, then the exponential function on the time scale $\mathbb{T}$ is define by

$$e_p(t, s) = \exp\left(\int_s^t \xi_{\mu(\tau)}(p(\tau))\Delta\tau\right) \text{ for all } s, t \in \mathbb{T},$$

where the cylinder transformation $\xi_h(z)$ is defined by $\xi_h(z) := \begin{cases} \frac{\mathrm{Log}(1+zh)}{h} & \text{if } h > 0, \\ z & \text{if } h = 0, \end{cases}$

1.3 Dynamic Inequalities

Lemma 1.36 (Gronwall-Bellman's Lemma, Bohner & Peterson (2001), page 257)**.** *Let* $y \in C_{\mathrm{rd}}(\mathbb{T}, \mathbb{R})$ *and* $k \in \mathcal{R}^{+}(\mathbb{T}, \mathbb{R})$, $k \geq 0$, $\alpha \in \mathbb{R}$. *Assume that* $y(t)$ *satisfies the inequality* $y(t) \leq \alpha + \int\limits_{t_0}^{t} k(s)y(s)\Delta s$, *for all* $t \in \mathbb{T}, t \geq t_0$. *Then,* $y(t) \leq \alpha e_{k(t)}(t, t_0)$ *holds for all* $t \in \mathbb{T}, t \geq t_0$.

Theorem 1.39 (Hölder's Inequality, Bohner & Peterson (2001), page 259)**.** *Let* $a, b \in \mathbb{T}$. *For rd-continuous functions* $f, g : (a, b) \to \mathbb{R}$, $p > 1$ *and* $\frac{1}{p} + \frac{1}{q} = 1$, *we have*

$$\int_a^b |f(t)g(t)|\Delta t \leq \left\{\int_a^b |f(t)|^p \Delta t\right\}^{\frac{1}{p}} \left\{\int_a^b |g(t)|^q \Delta t\right\}^{\frac{1}{q}},$$

1.4 Linear Dynamic Equation

Let $A : \mathbb{T}^\kappa \to \mathbb{R}^{n\times n}$ be rd-continuous and consider the n-dimensional linear dynamic equations $x^\Delta = A(t)x$ for all $t \in \mathbb{T}$.

Theorem 1.42 (Hilger (1990))**.** *Assume that $A(\cdot)$ is rd-continuous matrix valued function. Then, for each $t_0 \in \mathbb{T}^\kappa$, the initial value problem*

$$x^\Delta = A(t)x, \; x(t_0) = x_0 \tag{1.1}$$

has a unique solution $x(\cdot)$ defined on $t \geq t_0$. Moreover, if $A(\cdot)$ is regressive then this solution defines on $t \in \mathbb{T}^\kappa$.

The solution of Equation (1.1) is called *Cauchy operator*, or the *matrix exponential function* and denoted by $\Phi_A(t, t_0)$ or $\Phi(t, t_0)$.

Theorem 1.44 (Bohner & Peterson (2001), page 195)**.** *Let $A : \mathbb{T}^\kappa \to \mathbb{R}^{m\times m}$ be regressive and $f : \mathbb{T}^\kappa \times \mathbb{R}^m \to \mathbb{R}^m$ be rd-continuous. If $x(t)$, $t \geq t_0$, is a solution of dynamic equation $x^\Delta = A(t)x + f(t, x)$, $x(t_0) = x_0$, then we have*

$$x(t) = \Phi_A(t, t_0)x_0 + \int_{t_0}^{t} \Phi_A(t, \sigma(s)) f(s, x(s))\Delta s, \quad t \geq t_0.$$

1.5 Stability of Dynamic Equation

Let $\mathbb{T}$ be a time scale, $t_0 \in \mathbb{T}$. Consider dynamic equation of the form

$$x^\Delta = f(t, x), \quad x(t_0) = x_0 \in \mathbb{R}^m, \quad t \in \mathbb{T}, \tag{1.2}$$

where $f : \mathbb{T} \times \mathbb{R}^m \to \mathbb{R}^m$ is rd-continuous. If $f(t, 0) = 0$, then Equation (1.2) has the trivial solution $x \equiv 0$. Denote by $x(t; t_0, x_0)$ the solution of Cauchy problem (1.2).

Definition 1.45 (DaCunha (2005a), Hilger (1990))**.** The trivial solution $x \equiv 0$ of dynamic equation (1.2) is said to be exponentially stable if there exist a positive constant α with $-\alpha \in \mathcal{R}^+$ and a positive number $\delta > 0$ such that for each $t_0 \in \mathbb{T}$ there exists an $N = N(t_0) > 0$ for which, the solution of (1.2) with the initial condition $x(t_0) = x_0$ satisfies $\|x(t; t_0, x_0)\| \leq N\|x_0\|e_{-\alpha}(t, t_0)$, for all $t \geq t_0, t \in \mathbb{T}$ and $\|x_0\| < \delta$.

Definition 1.46 (Gard et.al. (2003))**.** The trivial solution $x \equiv 0$ of dynamic equation (1.2) is said to be exponentially stable if there exist a positive constant α and a positive number $\delta > 0$ such that for each $t_0 \in \mathbb{T}$, there exists an $N = N(t_0) > 0$ for which, the solution of (1.2) with the initial condition $x(t_0) = x_0$ satisfies $\|x(t; t_0, x_0)\| \leq N\|x_0\|e^{-\alpha(t-t_0)}$, for all $t \geq t_0, t \in \mathbb{T}$ and $\|x_0\| < \delta$.

If the constant N can be chosen independently of $t_0 \in \mathbb{T}$ then the solution $x \equiv 0$ of (1.2) is called uniformly exponentially stable.

Theorem 1.47 (Lan & Liem (2010))**.** *On the time scales with bounded graininess, Definition* 1.46 *is equivalent to Definition* 1.47.

CHAPTER 2

LYAPUNOV EXPONENTS FOR DYNAMIC EQUATIONS

In this chapter, we will study the first Lyapunov method for dynamic equations on time scales with a suitable approach. The content of chapter 2 is based on paper No.1 in list of the author's scientific works.

Since it is not able to define the logarithm function on time scales (Bohner (2005)), we use the oscillation of the ratio $\frac{|f(t)|}{e_\alpha(t,t_0)}$ as $t \to \infty$ in the parameter α to define Lyapunov exponent of the function f on a time scale with a certain parameter α.

Let $\mathbb{T}$ be unbounded above time scale, i.e., $\sup \mathbb{T} = \infty$, and the graininess $\mu(t)$ is bounded on $\mathbb{T}$, i.e., there exists a number $\mu^* = \sup_{t\in\mathbb{T}} \mu(t) < \infty$. This is equivalent to the existence of positive numbers m_1, m_2 such that for every element $t \in \mathbb{T}$, there exists a quantity that depends on t, $c = c(t) \in \mathbb{T}$, satisfying the condition $m_1 \le c - t < m_2$, see Pötzsche (2004). Furthermore, by definition, if $\alpha \in \mathbb{R} \cap \mathcal{R}^+$ then $\alpha > -\frac{1}{\mu(t)}$ for all $t \in \mathbb{T}$. Consequently, we have $\inf(\mathbb{R} \cap \mathcal{R}^+) = -\frac{1}{\mu^*}$, supplemented by $\frac{1}{0} = \infty$.

2.1 Lyapunov Exponent: Definition and Properties

2.1.1 Definition

Definition 2.1. *Lyapunov exponent* of the function f defined on time scale $\mathbb{T}_{t_0}$, valued in $\mathbb{K}$, is a real number $a \in \mathcal{R}^+$ such that for all arbitrary numbers $\varepsilon > 0$, we have

$$\lim_{t\to\infty} \frac{|f(t)|}{e_{a\oplus\varepsilon}(t,t_0)} = 0, \tag{2.1}$$

$$\limsup_{t\to\infty} \frac{|f(t)|}{e_{a\ominus\varepsilon}(t,t_0)} = \infty. \tag{2.2}$$

The Lyapunov exponent of function f is denoted by $\kappa_L[f]$.

If (2.1) is true for all $a \in \mathbb{R} \cap \mathcal{R}^+$ then we say by convention that f has *left extreme exponent*, $\kappa_L[f] = -\frac{1}{\mu^*} = \inf(\mathbb{R} \cap \mathcal{R}^+)$. If (2.2) is true for all $a \in \mathbb{R} \cap \mathcal{R}^+$, we say that the function f has *right extreme exponent*, $\kappa_L[f] = +\infty$. If $\kappa_L[f]$ is neither left extreme exponent nor right extreme exponent, then we call $\kappa_L[f]$ by *normal Lyapunov exponent*.

Lemma 2.2. *Let $f : \mathbb{T}_{t_0} \to \mathbb{K}$ be a function. Then, f has a normal Lyapunov exponent if and only if there exist two real numbers $\lambda, \gamma \in \mathcal{R}^+$ with $\lambda \neq \inf(\mathbb{R} \cap \mathcal{R}^+)$ such that*

$$\lim_{t\to\infty} \frac{|f(t)|}{e_\gamma(t,t_0)} = 0; \quad \limsup_{t\to\infty} \frac{|f(t)|}{e_\lambda(t,t_0)} = \infty. \tag{2.3}$$

Remark 2.3. i) In case $\mathbb{T} = \mathbb{R}$, Definition 2.1 leads to the classical one of Lyapunov exponent, i.e., $\kappa_L[f] = \chi[f] = \limsup\limits_{t\to\infty} \frac{\ln|f(t)|}{t}$.

ii) In case $\mathbb{T} = \mathbb{Z}$, we can see directly that $\ln(1 + \kappa_L[f]) = \limsup\limits_{n\to\infty} \frac{\ln|f(n)|}{n} = \chi[f]$.
Furthermore, the left extreme exponent is $\inf(\mathbb{R} \cap \mathcal{R}^+) = -1$.

2.1.2 Properties

We always suppose that $f, g : \mathbb{T}_{t_0} \to \mathbb{K}$ are the functions.

Lemma 2.4. *The following assertions hold true:*

i) $\kappa_L[|f|] = \kappa_L[f]$;

ii) $\kappa_L[0] = \inf(\mathbb{R} \cap \mathcal{R}^+)$ (*left extreme exponent*);

iii) $\kappa_L[cf] = \kappa_L[f]$, *where $c \neq 0$ is a constant;*

iv) *If $a \in \mathbb{R} \cap \mathcal{R}^+$ and* (2.1) *is satisfied for any $\varepsilon > 0$ then $\kappa_L[f] \leq a$. Similarly, if $a \in \mathbb{R} \cap \mathcal{R}^+$ and* (2.2) *holds for any $\varepsilon > 0$ then $\kappa_L[f] \geq a$;*

v) *If $|f(t)| \leq |g(t)|$ for all t large enough, then $\kappa_L[f] \leq \kappa_L[g]$;*

vi) *If f is bounded from above* (*resp. from below*) *then $\kappa_L[f] \leq 0$* (*resp. $\kappa_L[f] \geq 0$*). *As a consequence, if f is bounded then $\kappa_L[f] = 0$.*

Lemma 2.5. *For any $\lambda \in \mathcal{R} \cap \mathbb{C}$, the following assertions hold true.*

i) $\kappa_L[e_\lambda(\cdot, t_0)] = \kappa_L[e_{\widehat{\Re}\lambda}(\cdot, t_0)]$;

ii) $\kappa_L[e_\lambda(\cdot, t_0)]$ *does not depend on t_0;*

iii) *If $q(\cdot) \in \mathcal{R}^+$ then* $\kappa_L[e_q(\cdot, t_0)] \leq \limsup_{t\to\infty} q(t)$;

iv) $\kappa_L[e_\lambda(\cdot, t_0)] \leq \limsup_{t\to\infty} \widehat{\Re}\lambda(t) \leq |\lambda|$;

v) $\Re\lambda \leq \liminf_{t\to\infty} \widehat{\Re}\lambda(t) \leq \kappa_L[e_\lambda(\cdot, t_0)]$.

Lemma 2.7. $\kappa_L[f + g] \leq \max\{\kappa_L[f], \kappa_L[g]\}$ *and if $\kappa_L[f] \neq \kappa_L[g]$ then the equality holds.*

Lemma 2.9. $\kappa_L[fg] \leq \kappa_L[e_{\kappa_L[f] \oplus \kappa_L[g]}(\cdot, t_0)]$.

Definition 2.10. The function f is said to have exact Lyapunov exponent α if

$$\lim_{t\to\infty} \frac{|f(t)|}{e_{\alpha\oplus\varepsilon}(t,t_0)} = 0 \quad \text{and} \quad \lim_{t\to\infty} \frac{|f(t)|}{e_{\alpha\ominus\varepsilon}(t,t_0)} = \infty, \text{ for any } \varepsilon > 0.$$

Lemma 2.11. *If at least one of the functions f and g has exact Lyapunov exponent, then*

$$\kappa_L[fg] = \kappa_L[e_{\kappa_L[f]\oplus\kappa_L[g]}(\cdot, t_0)].$$

Remark 2.12. In case both f and g have exact Lyapunov exponents, then so does the function fg, and $\kappa_L[fg] = \kappa_L[e_{\kappa_L[f]\oplus\kappa_L[g]}(\cdot, t_0)]$. Generally, if all of the functions $f_1, f_2, ..., f_m$ have exact Lyapunov exponents then the function $f_1 f_2 \cdots f_m$ does, too, and $\kappa_L[f_1 f_2 \cdots f_m] = \kappa_L[e_{\kappa_L[f_1]\oplus\kappa_L[f_2]\oplus\cdots\oplus\kappa_L[f_m]}(\cdot, t_0)]$.

Remark 2.13. i) In case $\mathbb{T} = \mathbb{R}$, $\kappa_L[fg] \le \kappa_L[e_{\kappa_L[f]\oplus\kappa_L[g]}(\cdot, t_0)] = \kappa_L[f] + \kappa_L[g]$.

ii) In case $\mathbb{T} = \mathbb{Z}$, $\kappa_L[fg] \le \kappa_L[e_{\kappa_L[f]\oplus\kappa_L[g]}(\cdot, t_0)] = \kappa_L[f] + \kappa_L[g] + \kappa_L[f]\kappa_L[g]$ (or equivalently, $\chi[fg] \le \chi[f] + \chi[g]$).

2.2 Lyapunov Exponents of Solutions of Linear Equation

2.2.1 Lyapunov Spectrum of Linear Equation

Consider the linear equation

$$x^\Delta = A(t)x, \tag{2.4}$$

where $A(t)$ is a regressive and rd-continuous $n \times n$-matrix on time scale $\mathbb{T}$. It is known that Eq. (2.4) with the initial value $x(t_0) = x_0$ has an unique solution $x(t) = x(t; t_0, x_0)$ on $\mathbb{T}$.

Theorem 2.15. *Let $\mathcal{M} = \limsup_{t\to\infty} \|A(t)\|$. If $x(\cdot)$ is a nontrivial solution of Eq.* (2.4), *then $\kappa_L[x(\cdot)] \le \mathcal{M}$. Furthermore, if* $\limsup_{t\to\infty} \mu(t) < \frac{1}{\mathcal{M}}$, *then the appreciation $-\mathcal{M} \le \kappa_L[x(\cdot)] \le \mathcal{M}$ holds.*

In case $\mathbb{T} = \mathbb{R}$, we get a popular inequality $-\mathcal{M} \le \kappa_L[x(\cdot)] = \chi[x(\cdot)] \le \mathcal{M}$.

Definition 2.17. The set of all finite Lyapunov exponents of solutions of Eq. (2.4) is called the *Lyapunov spectrum* of this equation.

Theorem 2.18. *The Lyapunov spectrum of Eq.* (2.4) *has n distinct values at most.*

2.2.2 Lyapunov Inequality

Let $\{x_1(t), x_2(t), ..., x_n(t)\}$ be a system of regular fundamental solutions of Eq. (2.4), i.e., the system of these solutions has properties: The Lyapunov exponent of solutions combined from some arbitrary solutions of this system will be equal to the Lyapunov exponent of a solution attending in the combination. In other words, if

$$x(t) = k_1 x_1(t) + k_2 x_2(t) + \cdots + k_n x_n(t),$$

then $\kappa_L[x(\cdot)] = \kappa_L[x_i(\cdot)]$ with some $i \in \{1,\dots,n\}$.

Denote by $S = \{\alpha_1, \alpha_2, \dots, \alpha_n | \alpha_1 \leq \alpha_2 \leq \dots \leq \alpha_n\}$ the set of Lyapunov spectrum of Eq. (2.4). In addition, we suppose that $\alpha_i \in \mathbb{R} \cap \mathcal{R}^+$, for all $i = 1, 2, ..., n$.

Theorem 2.19 (Lyapunov's Inequality). $\kappa_L[e_\alpha(\cdot, t_0)] \leq \kappa_L[e_{\alpha_1 \oplus \alpha_2 \oplus \dots \oplus \alpha_n}(\cdot, t_0)]$.

Note that, the case $\mathbb{T} = \mathbb{R}$, we have

$$\kappa_L[e_\alpha(\cdot, t_0)] = \limsup_{t\to\infty} \frac{1}{t - t_0} \int_{t_0}^{t} (\text{trace}\, A(s))ds,$$

and

$$\kappa_L\left[e_{\alpha_1 \oplus \dots \oplus \alpha_n}(\cdot, t_0)\right] = \alpha_1 + \dots + \alpha_n.$$

Thus, we get the Lyapunov inequality for ordinary differential equations in Malkin (1958).

We consider Eq. (2.4), where $A(t) \equiv A$ is a constant and regressive $n \times n$-matrix. Let $\lambda_i, i = 1, 2, ..., n$ be the eigenvalues of matrix A. It is easy to verify that

$$\alpha(t) = \lambda_1 \oplus \lambda_2 ... \oplus \lambda_n(t). \tag{2.5}$$

Theorem 2.22. *If for any eigenvalue λ_i of matrix A, the exponential function $e_{\lambda_i}(\cdot, t_0)$ has the exact Lyapunov exponent, then $\kappa_L[e_\alpha(\cdot, t_0)] = \kappa_L[e_{\alpha_1 \oplus \alpha_2 \oplus \dots \oplus \alpha_n}(\cdot, t_0)]$, where $\alpha_i = \kappa_L[e_{\lambda_i}(\cdot, t_0)]$, $i = 1, 2, ..., n$.*

2.3 Lyapunov Spectrum and Stability of Linear Equation

Consider the equation

$$x^\Delta = A(t)x, \tag{2.6}$$

where $A(t)$ is a regressive, rd-continuous $n \times n$-matrix, $\|A(t)\| \leq \mathcal{M}$, for all $t \in \mathbb{T}_\tau$.

Theorem 2.24. *Consider Eq.* (2.6) *with the stated conditions on $A(\cdot)$. Then,*

i) *Eq.* (2.6) *is exponentially asymptotically stable if and only if there exists a constant $\alpha > 0$ with $-\alpha \in \mathcal{R}^+$ such that for every $t_0 \in \mathbb{T}_\tau$, there is a number $N = N(t_0) \geq 1$ such that*

$$\|\Phi_A(t, t_0)\| \leq N e_{-\alpha}(t, t_0) \text{ for all } t \geq t_0, t \in \mathbb{T}_\tau.$$

ii) *Eq.* (2.6) *is uniformly exponentially asymptotically stable if and only if there exist constants $\alpha > 0$, $N \geq 1$ with $-\alpha \in \mathcal{R}^+$ such that*

$$\|\Phi_A(t, t_0)\| \leq N e_{-\alpha}(t, t_0) \text{ for all } t \geq t_0, t, t_0 \in \mathbb{T}_\tau.$$

We give the spectral condition for exponential stability.

Theorem 2.25. *Let $-\alpha := \max S$, where S is the set of Lyapunov spectrum of Eq.* (2.6). *Then, Eq.* (2.6) *is exponentially asymptotically stable if and only if $\alpha > 0$.*

We now consider the following equation

$$x^{\Delta} = Ax. \tag{2.7}$$

where A is a regressive constant matrix. Denote the set of all eigenvalues of the matrix A by $\sigma(A)$. From the regressivity of the matrix A, it follows that $\sigma(A) \subset \mathcal{R}$.

Theorem 2.26. *If Eq.* (2.7) *is exponentially asymptotically stable then* $\kappa_L[e_\lambda(\cdot, t_0)] < 0$, *for all* $\lambda \in \sigma(A)$. *In addition, suppose that every eigenvalue* $\lambda \in \sigma(A)$ *is uniformly regressive. Then, the assumption* $\kappa_L[e_\lambda(\cdot, t_0)] < 0$ *implies that Eq.* (2.7) *is exponentially asymptotically stable.*

Corollary 2.27. *If for any eigenvalue* $\lambda \in \sigma(A)$ *we have* $\Im\lambda \neq 0$ *and* $\kappa_L[e_\lambda(\cdot, t_0)] < 0$, *then Eq.* (2.7) *is exponentially asymptotically stable.*

Theorem 2.28. *Suppose that* $\limsup_{t\to\infty} \widehat{\Re}\lambda(t) < 0$ *for all* $\lambda \in \sigma(A)$. *Then, Eq.* (2.7) *is exponentially asymptotically stable.*

Corollary 2.29. *If* $\sigma(A) \subset (-\infty, 0) \cap \mathcal{R}^+$ *then Eq.* (2.7) *is exponentially asymptotically stable.*

Example 2.30. Considering Eq. $x^{\Delta}(t) = Ax(t)$ on time scale $\mathbb{T} = \cup_{k=0}^{\infty}[2k, 2k+1]$, with

$$A = \frac{1}{24}\begin{pmatrix} -24 & 0 & 48 \\ 1 & -24 & 24 \\ 33 & -72 & -48 \end{pmatrix}.$$

It is clear that

$$\mu(t) = \begin{cases} 0 & \text{if } t \in \cup_{k=0}^{\infty}[2k, 2k+1), \\ 1 & \text{if } t \in \cup_{k=0}^{\infty}\{2k+1\}, \end{cases}$$

the left extreme exponent is -1. Further, $\sigma(A) = \left\{-2, -1 + \frac{1}{2}i, -1 - \frac{1}{2}i\right\}$ and all $\lambda \in \sigma(A)$ are uniformly regressive.

i) In case $\lambda_1 = -2$, $t \in [2k, 2k+1]$, we have $\kappa_L[e_{-2}(\cdot, 0)] \leq \kappa_L[e_{-\frac{1}{2}}(\cdot, 0)] = -\frac{1}{2} < 0$.

ii) In case $\lambda_2 = -1 + \frac{i}{2}$, we have $\kappa_L[e_{\lambda_2}(\cdot, 0)] \leq \limsup_{t\to\infty} \widehat{\Re}\lambda_2(t) = \frac{1}{\sqrt{2}} - 1 < 0$.

iii) In case $\lambda_3 = -1 - \frac{i}{2}$, we get $\kappa_L[e_{\lambda_3}(\cdot, 0)] \leq \limsup_{t\to\infty} \widehat{\Re}\lambda_3(t) = \frac{1}{\sqrt{2}} - 1 < 0$.

Therefore, by Theorem 2.23, the equation is exponentially asymptotically stable.

Make a note that the equation $x^{\Delta}(t) = -2x(t)$, $t \in \mathbb{T} = \cup_{k=0}^{\infty}[2k, 2k+1]$ is exponentially asymptotically stable, meanwhile

$$\limsup_{t\to\infty} \widehat{\Re}(-2)(t) = 0.$$

This indicates that, in general, the inverse of Theorem 2.23 is not true.

CHAPTER 3

BOHL EXPONENTS FOR IMPLICIT DYNAMIC EQUATIONS

Consider linear time-varying IDE of the form

$$E_\sigma(t)x^\Delta(t) = A(t)x(t), \quad t \geq 0, \tag{3.1}$$

where $E_\sigma(\cdot), A(\cdot)$ are continuous matrix funtions, $E_\sigma(\cdot)$ is supposed to be singular. If Eq. (3.1) is subject to an external force $f(t)$, then it becomes

$$E_\sigma(t)x^\Delta(t) = A(t)x(t) + f(t), \quad t \geq 0. \tag{3.2}$$

We will introduce the concept of Bohl exponent of linear time-varying IDEs with index-1 and investigate the relation between the exponential stability and Bohl exponent as well as the robustness of Bohl exponent. The content of Chapter 3 is based on the papers No.2 and No.3 in list of the author's works.

3.1 Linear Implicit Dynamic Equations with index-1

Consider linear time-varying IDE (3.2) for all $t \geq a > 0$, where A, E_σ are in $L_\infty^{\text{loc}}(\mathbb{T}_a; \mathbb{K}^{n\times n})$. Assume that rank $E = r$, $1 \leq r < n$, for all $t \in \mathbb{T}_a$ and $\ker E$ is smooth in the sense that there exists a projector Q onto $\ker E$ such that Q is continuously differentiable for all $t \in (a, \infty)$, $Q^2 = Q$ and $Q^\Delta \in L_\infty^{\text{loc}}(\mathbb{T}_a; \mathbb{K}^{n\times n})$. Set $P = I - Q$, P is a projector along $\ker E$, $EP = E$. Then, Eq. (3.2) can be rewritten

$$E_\sigma(t)(Px)^\Delta(t) = \bar{A}(t)x(t) + f(t),\ t \geq a, \bar{A} := A + E_\sigma P^\Delta \in L_\infty^{\text{loc}}(\mathbb{T}_a; \mathbb{K}^{n\times n}). \tag{3.3}$$

Let H be a function taking values in the group $\mathrm{Gl}(\mathbb{R}^n)$ such that $H|_{\ker E_\sigma}$ is an isomorphism between $\ker E_\sigma$ and $\ker E$. Set $G := E_\sigma - \bar{A}HQ_\sigma$, and $S := \{x : Ax \in \operatorname{im} E_\sigma\}$.

Lemma 3.2 (Du et al. (2007)). *Suppose that the matrix G is nonsingular.*

i) $P_\sigma = G^{-1}E_\sigma$;   ii) $G^{-1}\bar{A}HQ_\sigma = -Q_\sigma$;

iii) $\widetilde{Q} := -HQ_\sigma G^{-1}\bar{A}$ *is the projector onto* $\ker E$ *along to* S, $\widetilde{Q}$ *is a canonical projector;*

iv) *If* $\widehat{Q}$ *is a projector onto* $\ker E$, $\widehat{P} = I - \widehat{Q}$, *then* $P_\sigma G^{-1}\bar{A} = P_\sigma G^{-1}\bar{A}\widehat{P}$, $Q_\sigma G^{-1}\bar{A} = Q_\sigma G^{-1}\bar{A}\widehat{P} - H^{-1}\widehat{Q}$;

v) *The matricies $P_\sigma G^{-1}$, $HQ_\sigma G^{-1}$ does not depend on the choice of H and Q.*

Definition 3.4. The IDE (3.2) is said to be index-1 tractable on $\mathbb{T}_a$ if $G(t)$ is invertible for almost $t \in \mathbb{T}_a$ and $G^{-1} \in L^{\text{loc}}_\infty(\mathbb{T}_a; \mathbb{K}^{n\times n})$.

Let $J \subset \mathbb{T}$ be an interval. We denote the set

$$C^1(J, \mathbb{K}^n) := \{x(\cdot) \in C_{rd}(J, \mathbb{K}^n) : P(t)x(t) \text{ is delta differentiable, almost } t \in J\}.$$

Definition 3.6. The function x is said to be a solution of Eq. (3.2) (having index-1) on the interval J if $x \in C^1(J, \mathbb{K}^n)$ and satisfies Eq. (3.2) for almost $t \in J$.

Multiplying both sides of Eq. (3.3) by $P_\sigma G^{-1}$ and $Q_\sigma G^{-1}$ and using variable changes $u := Px$ and $v := Qx$, Eq. (3.3) is decomposed into two sub-equations

$$u^\Delta = (P^\Delta + P_\sigma G^{-1}\bar{A})u + P_\sigma G^{-1} f, \tag{3.4}$$

$$v = HQ_\sigma G^{-1}\bar{A}u + HQ_\sigma G^{-1} f, \tag{3.5}$$

(3.4) is called the delta-differential part and (3.5) the algebraic one. We can solve u from Eq. (3.4), get v from (3.5), and $x = u + v$. The solution of (3.2) is

$$x(t) = \Phi(t, t_0)P(t_0)x_0 + \int_{t_0}^{t} \Phi(t, \sigma(s))P_\sigma(s)G^{-1}(s)f(s)\Delta s + H(t)Q_\sigma(t)G^{-1}(t)f(t).$$

3.2 Stability of IDEs under non-Linear Perturbations

Let $a \in \mathbb{T}$ be a fixed point. In case the external force $f(t) := F(t, x(t))$, where F is a certain function defined on $\mathbb{T}_a \times \mathbb{R}^n$, then Eq. (3.2) is rewritten as follows

$$E_\sigma(t)x^\Delta(t) = A(t)x(t) + F(t, x(t)),\ t \geq a. \tag{3.6}$$

Let $F(t, 0) = 0$ for all $t \in \mathbb{T}_a$. So, Eq. (3.6) has the trivial solution $x(t) \equiv 0$. As before, denoting $u = Px$ and $v = Qx$ comes to

$$u^\Delta = (P^\Delta + P_\sigma G^{-1}\bar{A})u + P_\sigma G^{-1}F(t, u + v), \tag{3.7}$$

$$v = HQ_\sigma G^{-1}\bar{A}u + HQ_\sigma G^{-1}F(t, u + v). \tag{3.8}$$

Assume that $HQ_\sigma G^{-1}F(t, \cdot)$ is Lipschitz continuous with Lipschitz coefficient $\gamma_t < 1$, i.e., $\|HQ_\sigma G^{-1}F(t, y) - HQ_\sigma G^{-1}F(t, z)\| \leq \gamma_t\|y - z\|$, $\forall t \geq a$. Since $HQ_\sigma G^{-1}$ does not depend on the choice of H and Q, the Lipschitz property of $HQ_\sigma G^{-1}F(t, \cdot)$ does, too. Fix $u \in \mathbb{R}^n$ and choose $t \in \mathbb{T}_a$, we consider a mapping $\Gamma_t : \operatorname{im} Q(t) \to \operatorname{im} Q(t)$ defined by $\Gamma_t(v) := H(t)Q_\sigma(t)G^{-1}(t)\bar{A}(t)u + H(t)Q_\sigma(t)G^{-1}(t)F(t, u + v)$. It is easy to see that $\|\Gamma_t(v) - \Gamma_t(v')\| \leq \gamma_t\|v - v'\|$ for any $v, v' \in \operatorname{im} Q(t)$. Since $\gamma_t < 1$, Γ_t is a contractive mapping. Hence, by the Fixed Point Theorem, there exists a mapping $g_t : \operatorname{im} P(t) \to \operatorname{im} Q(t)$ satisfying

$$g_t(u) = H(t)Q_\sigma(t)G^{-1}(t)\big(\bar{A}(t)u + F(t, u + g_t(u))\big). \tag{3.9}$$

Denoted by $\beta_t := \|H(t)Q_\sigma(t)G^{-1}(t)\bar{A}(t)\|$, we get $\|g_t(u) - g_t(u')\| \leq \frac{\gamma_t+\beta_t}{1-\gamma_t}\|u - u'\|$. Thus, g_t is Lipschitz continuous with Lipschitz constant $L_t = \frac{\gamma_t+\beta_t}{1-\gamma_t}$. Substituting $v = g_t(u)$ into (3.7) obtains

$$u^\Delta = (P^\Delta + P_\sigma G^{-1}\bar{A})u + P_\sigma G^{-1}F\big(t, u + g_t(u)\big). \tag{3.10}$$

We can solve $u(t)$ from Eq. (3.10). Therefore, the unique solution of (3.6) is

$$x(t) = u(t) + g_t(u(t)),\ t \in \mathbb{T}_a. \tag{3.11}$$

Theorem 3.10. *Assume that Equation* (3.1) *is of* index-1, *exponential stable and*

i) $L = \sup_{t\in\mathbb{T}_a} L_t < \infty$, *and*

ii) *the function* $P_\sigma(t)G^{-1}(t)F(t,x)$ *is Lipschitz continuous with Lipschitz constant* k_t, *such that one of the following conditions hold*

   a) $N = \displaystyle\int_a^\infty \frac{k_t}{1-\alpha\mu(t)}\Delta t < \infty.$

   b) $\limsup_{t\to\infty} k_t(1 + L_t) = \delta < \frac{\alpha}{LM}$, *with* α, M *are positive and* $-\alpha \in \mathcal{R}^+$.

Then, there exist the constants $K > 0$ *and positively regressive* $-\alpha_1$ *such that*

$$\|x(t)\| \leq Ke_{-\alpha_1}(t,s)\|P(s)x(s)\|,$$

for all $t \geq s \geq a$, *where* $x(\cdot)$ *is a solution of* (3.6). *That is, the perturbed equation* (3.6) *preserves the exponential stability.*

Next, we prove the Bohl-Perron Theorem for linear IDEs, i.e., investigate the relation between the boundedness of solutions of non-homogenous Eq. (3.2) and the exponential stability of IDE (3.1).

Note that, in solving Eq. (3.2), the function f is split into two components $P_\sigma G^{-1}f$ and $HQ_\sigma G^{-1}f$. Therefore, for any $t_0 \in \mathbb{T}_a$ we consider f as an element of the set

$$L(t_0) = \left\{ \begin{array}{r} f \in C\left([t_0,\infty], \mathbb{R}^n\right) : \sup_{t\geq t_0} \|H(t)Q_\sigma(t)G^{-1}(t)f(t)\| < \infty \\ \text{and } \sup_{t\geq t_0} \|P_\sigma(t)G^{-1}(t)f(t)\| < \infty \end{array} \right\}.$$

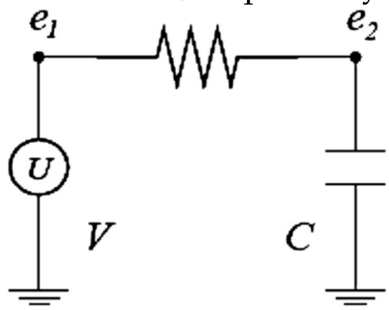
It is easy to see that $L(t_0)$ is a Banach space eqiupped with the norm

$$\|f\| = \sup_{t\geq t_0} \left(\|P_\sigma(t)G^{-1}(t)f(t)\| + \|H(t)Q_\sigma(t)G^{-1}(t)f(t)\|\right).$$

Denote by $x(t,s,f)$ the solution, associated with f, of Eq. (3.2) with the initial condition $P(s)x(s,s) = 0$. For notational convenience, we will write $x(t,s)$ or $x(t)$ for $x(t,s,f)$ if there is no confusion.

Theorem 3.14. *All solutions of Cauchy problem* (3.2) *with the initial condition* $P(t_0)x(t_0) = 0$, *associated with an arbitrary function* f *in* $L(t_0)$, *are bounded if and only if the* index-1 *IDE* (3.1) *is exponentially stable.*

Remark 3.15. The above results extended the Bohl-Perron type stability theorem with bounded input/output for differential and difference equations, for differential algebraic, and for implicit difference equations, in case $\mathbb{T} = \mathbb{R}$ or $\mathbb{T} = \mathbb{Z}$, respectively.



Example 3.16. Consider the simple circuit on time scales consists of a voltage source $v_V = v(t)$, a resistor with conductance R and a capacitor with capacitance $C > 0$. As in Tischendorf (2000), this model can be written in the form $E_\sigma x^\Delta = Ax + f$, with $x = \begin{bmatrix} e_1 & e_2 & i_v \end{bmatrix}^T$, $f = \begin{bmatrix} 0 & 0 & v \end{bmatrix}^T$,

$E_\sigma = \begin{bmatrix} 0 & 0 & 0 \\ 0 & C & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $A = \begin{bmatrix} -R & R & 1 \\ R & -R & 0 \\ 1 & 0 & 0 \end{bmatrix}$. It is easy to choose P, and $H = I$. We compute G^{-1}. This implies that $\|f\| = \sqrt{1 + \frac{R^2(C^2+1)}{C^2}}\|v\|$. On the other hand, the spectral set $\sigma(E_\sigma, A) = \left\{\frac{-R}{C}\right\}$. Therefore, if $1 - \frac{\mu(t)R}{C} > 0$, or equivalently $\frac{-R}{C} \in \mathcal{R}^+$ then the homogenous equation $E_\sigma x^\Delta = Ax$ is exponentially stable. By Theorem 3.14, if v is bounded then e_1, e_2, i_v are bounded.

3.3 Bohl Exponent for IDEs

3.3.1 Bohl Exponent: Definition and Property

Definition 3.17. Let the IDE (3.1) be index-1, $\Phi(t,s)$ be its Cauchy operator. Then, the (upper) Bohl exponent of IDE (3.1) is defined by

$$\kappa_{\mathcal{B}}(E, A) = \inf\{\alpha \in \mathbb{R}; \exists M_\alpha > 0 : \|\Phi(t,s)\| \le M_\alpha e_\alpha(t,s), \forall t \ge s \ge t_0\}.$$

When $\kappa_{\mathcal{B}}(E, A) = -\frac{1}{\mu^*}$ or $\kappa_{\mathcal{B}}(E, A) = +\infty$ we call Bohl exponent of IDE (3.1) is extreme. In case $\mathbb{T} = \mathbb{R}$ (or resp. $\mathbb{T} = h\mathbb{Z}$), we come to the classical definition of Bohl exponents, and the extreme exponents may be $\pm\infty$ (resp. $-\frac{1}{h}$ or $+\infty$). Further,

Proposition 3.18. *If $\alpha = \kappa_{\mathcal{B}}(E, A)$ is not extreme then for any $\varepsilon > 0$ we have*

i) $\displaystyle\lim_{\substack{t-s\to\infty \\ s\to\infty}} \frac{\|\Phi(t,s)\|}{e_{\alpha\oplus\varepsilon}(t,s)} = 0$ ii) $\displaystyle\limsup_{\substack{t-s\to\infty \\ s\to\infty}} \frac{\|\Phi(t,s)\|}{e_{\alpha\ominus\varepsilon}(t,s)} = \infty.$

Example 3.20. Set $\mathbb{T} = \bigcup_{k=0}^{\infty} \{3k\} \bigcup_{k=0}^{\infty} [3k+1, 3k+2]$, and consider Equation (3.1) with

$$E(t) = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},\ A(t) = \begin{bmatrix} p(t) & p(t) & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},\ \text{and } p(t) = \begin{cases} -\frac{1}{4} & \text{if } t = 3k, \\ -\frac{1}{2} & \text{if } t \in [3k+1, 3k+2]. \end{cases}$$

We can choose $P, H = I$ and compute $\widetilde{P}, \Phi_0(t,s), \Phi(t,s)$... and $\kappa_{\mathcal{B}}(E, A) = -\alpha$.

Theorem 3.23. *The following statements are equivalent:*

i) *The IDE* (3.1) *is exponentially stable;* ii) *The Bohl exponent* $\kappa_{\mathcal{B}}(E, A)$ *is negative;*

iii) *The Bohl exponent* $\kappa_{\mathcal{B}}(E, A)$ *is finite and for any* $p > 0$*, there exists a positive constant* K_p *such that* $\int_s^\infty \|\Phi(t, s)\|^p \Delta t \le K_p$, $\forall t \ge s \ge t_0$;

iv) *All solutions of the Cauchy problem* (3.2) *with the initial condition* $P(t_0)x(t_0) = 0$*, associated with* f *in* $L(t_0)$ *are bounded.*

3.3.2 Robustness of Bohl Exponent

Suppose that $\Sigma(\cdot) \in \mathbb{R}^{n\times n}$ is a continuous matrix function. We consider the perturbed equation

$$E_\sigma(t)x^\Delta(t) = (A(t) + \Sigma(t))x(t),\ \forall t \ge t_0. \tag{3.12}$$

It is easy to see that, Eq. (3.12) is equivalent to

$$E_\sigma(t)(Px)^\Delta(t) = (\bar{A}(t) + \Sigma(t))x(t),\ \forall t \ge t_0. \tag{3.13}$$

Eq. (3.12) is a special case of (3.6) with $F(t, x) = \Sigma(t)x$. Let perturbation Σ be sufficiently small such that

$$\sup_{t\ge t_0} \|\Sigma(t)\| < \Big(\sup_{t\ge t_0} \|HQ_\sigma G^{-1}(t)\|\Big)^{-1}. \tag{3.14}$$

By using (3.14) and the relation $(I - \Sigma HQ_\sigma G^{-1})^{-1}G_\Sigma = G$, where $G_\Sigma := E_\sigma - (\bar{A} + \Sigma)HQ_\sigma$, it is easy to see that G_Σ is invertible if only if so is G. This means that Eq. (3.2) is index-1 if and only if Eq. (3.13) is, too. By the same argument as before, we can solve Eq. (3.13). Indeed, since the function $HQ_\sigma G^{-1}\Sigma(t)x$ is Lipschitz continuous with Lipschitz coefficient $\gamma_t = \|HQ_\sigma G^{-1}\Sigma(t)\| < 1$, the function g_t defined by (3.9) becomes $g_t(u) = (I - HQ_\sigma G^{-1}\Sigma(t))^{-1}HP_\sigma G^{-1}(\bar{A} + \Sigma)(t)u$. Then the solution of (3.13) is $x(t, s) = u(t, s) + g_t(u(t, s))$, where $u(t, s)$ is the solution of the IVP

$$\begin{cases} u^\Delta = (P^\Delta + P_\sigma G^{-1}\bar{A})u + P_\sigma G^{-1}\Sigma(u + g_t(u)), \\ u(s, s) = P(s)x_0. \end{cases}$$

Theorem 3.26. *Let* $P_\sigma G^{-1}$ *and* $HQ_\sigma G^{-1}$ *be bounded above. Then, for any* $\varepsilon > 0$ *there exists a number* $\delta = \delta(\varepsilon) > 0$ *such that the inequality* $\limsup_{t\to\infty} \|\Sigma(t)\| \le \delta$ *implies* $\kappa_{\mathcal{B}}(E, A + \Sigma) \le \kappa_{\mathcal{B}}(E, A) + \varepsilon$.

We now consider equation $E_\sigma(t)x^\Delta(t) = A(t)x(t)$, for all $t \ge t_0$, subject to two-side perturbations of the form

$$\big(E_\sigma(t) + F_\sigma(t)\big)x^\Delta(t) = \big(A(t) + \Sigma(t)\big)x(t),\ \forall t \ge t_0, \tag{3.15}$$

where $F_\sigma(t)$ and $\Sigma(t)$ are perturbation matrices, and $\ker(E_\sigma + F_\sigma) = \ker E_\sigma$. We can prove that, Eq. (3.15) is equivalent to $E_\sigma(t)x^\Delta(t) = \big(A(t) + \bar{\Sigma}(t)\big)x(t)$, $\forall t \ge t_0$.

Theorem 3.27. *Let* $P_\sigma G^{-1}$ *and* $HQ_\sigma G^{-1}$ *be bounded above. Then, for any* $\varepsilon > 0$ *there exists a number* $\delta = \delta(\varepsilon) > 0$ *such that the inequality* $\limsup_{t\to\infty} \|\bar{\Sigma}(t)\| \le \delta$ *implies* $\kappa_{\mathcal{B}}(E + F, A + \bar{\Sigma}) \le \kappa_{\mathcal{B}}(E, A) + \varepsilon$.

CHAPTER 4

STABILITY RADIUS FOR IMPLICIT DYNAMIC EQUATIONS

We will consider the robust stability of the system of linear time-varying IDE on time scales,

$$E_\sigma(t)x^\Delta(t) = A(t)x(t) + f(t), \quad t \geq t_0, \tag{4.1}$$

where $E_\sigma(\cdot) \in L_\infty^{\mathrm{loc}}(\mathbb{T};\mathbb{K}^{n\times n})$ is supposed to be singular for all $t \in \mathbb{T}$, $t \geq t_0$. The matrix $A(\cdot) \in L_\infty^{\mathrm{loc}}(\mathbb{T};\mathbb{K}^{n\times n})$, and $\ker A(\cdot)$ is absolutely continuous. The corresponding homogeneous equation is

$$E_\sigma(t)x^\Delta(t) = A(t)x(t), \quad t \geq t_0, \tag{4.2}$$

The content of Chapter 4 is based on the paper No.2 in list of the author's works.

Let X, Y be the finite-dimensional vector spaces. For every $p \in \mathbb{R}$, $1 \leq p < \infty$ and $s < t$, $s, t \in \mathbb{T}_a$, denote by $L_p([s,t];X)$ the space of measurable functions f on the interval $[s,t]$ equipped with the norm $\|f\|_p = \|f\|_{L_p([s,t];X)} := \left(\int_s^t \|f(\tau)\|^p \Delta\tau\right)^{\frac{1}{p}} < \infty$, and by $L_\infty([s,t];X)$ the space of measurable and essentially bounded functions f equipped with the norm $\|f\|_\infty = \|f\|_{L_\infty([s,t];X)} := \Delta\text{-}\operatorname{esssup}_{\tau\in[s,t]} \|f(\tau)\|$. We also consider the spaces $L_p^{\mathrm{loc}}(\mathbb{T}_a;X)$, $L_\infty^{\mathrm{loc}}(\mathbb{T}_a;X)$, which contain all functions f restricted on $[s,t]$, $f|_{[s,t]}$, are in $L_p([s,t];X)$, $L_\infty([s,t];X)$, respectively, for every $s,t \in \mathbb{T}_a$, $a \leq s < t < \infty$. For $\tau \geq a, \tau \in \mathbb{T}$, the operator of truncation π_τ at τ on the space $L_p(\mathbb{T}_a;X)$ is defined by

$$\pi_\tau(u)(t) := \begin{cases} u(t), & t \in [a,\tau], \\ 0, & t > \tau. \end{cases}$$

Denote by $\mathcal{L}(L_p(\mathbb{T}_a;X), L_p(\mathbb{T}_a;Y))$ for the Banach space of linear bounded operators Σ from $L_p(\mathbb{T}_a;X)$ to $L_p(\mathbb{T}_a;Y)$ and the corresponding norm is defined by

$$\|\Sigma\| := \sup_{x\in L_p(\mathbb{T}_a;X),\ \|x\|=1} \|\Sigma x\|_{L_p(\mathbb{T}_a;Y)}.$$

The operator $\Sigma \in \mathcal{L}(L_p(\mathbb{T}_a;X), L_p(\mathbb{T}_a;Y))$ is called to be *causal* if it satisfies

$$\pi_t\Sigma\pi_t = \pi_t\Sigma, \ \text{for every } t \geq a.$$

4.1 Stability of IDEs under Causal Perturbations

Consider the linear time-varying implicit dynamic equation (4.1), for all $t \geq a$, and the corresponding homogeneous equation

$$E_\sigma(t)x^\Delta(t,t_0) = A(t)x(t,t_0), t \geq a \tag{4.3}$$

with initial condition $P(t_0)(x(t_0,t_0) - x_0) = 0$.

Let $P(t), Q(t)$ be the projectors in Chapter 3, Eq. (4.1) comes to the form

$$E_\sigma(t)(Px)^\Delta(t) = \bar{A}(t)x(t) + f(t),\ t \geq a,\ \bar{A} := A + E_\sigma P^\Delta \in L^{\text{loc}}_\infty(\mathbb{T}_a;\mathbb{K}^{n\times n}) \tag{4.4}$$

Assumption 4.1. *The IDE* (4.3) *is of* index-1 *and uniformly exponential stable in the sense that there exist numbers* $M > 0, \omega > 0$ *such that* $-\omega$ *is positively regressive and*

$$\|\Phi(t,s\| \leq Me_{-\omega}(t,s),\ t \geq s,\ t,s \in \mathbb{T}_a.$$

Assumption 4.2. *There exists a bounded, smooth projector* $Q(t)$ *onto* $\ker E(t)$ *such that the terms* $P_\sigma G^{-1}$ *and* $HQ_\sigma G^{-1}$ *are essentially bounded on* $\mathbb{T}_a$*.*

We consider Eq. (4.3) subject to structured perturbations of the form

$$E_\sigma(t)x^\Delta(t) = A(t)x(t) + B(t)\Sigma\big(C(\cdot)x(\cdot)\big)(t),\ t \in \mathbb{T}_a, \tag{4.5}$$

where $B \in L_\infty(\mathbb{T}_a;\mathbb{K}^{n\times m})$ and $C \in L_\infty(\mathbb{T}_a;\mathbb{K}^{q\times n})$ are given matrices defining the structure of perturbations, $\Sigma : L_p(\mathbb{T}_a;\mathbb{K}^q) \to L_p(\mathbb{T}_a;\mathbb{K}^m)$ is an unknown disturbance operator supposed to be linear, causal. Therefore, with perturbation Σ, Eq. (4.5) becomes an implicit functional DAE.

We define the linear operator $\widetilde{G}$ from $L^{\text{loc}}_p(\mathbb{T}_a;\mathbb{K}^n)$ to $L^{\text{loc}}_p(\mathbb{T}_a;\mathbb{K}^n)$ which written formally by $\widetilde{G} = (I - B\Sigma CHQ_\sigma G^{-1})G$.

Definition 4.3. Implicit functional differential-algebraic equation (4.5) is said to be of index-1, in the generalized sense, if for any $T > a$, the operator $\widetilde{G}$ restricted to $L_p([a,T];\mathbb{K}^n)$ has the bounded inverse operator $\widetilde{G}^{-1}$.

For any $t_0 \in \mathbb{T}_a$, we set up Cauchy problem for Eq. (4.5)

$$\begin{cases} E_\sigma(t)x^\Delta(t) = A(t)x(t) + B(t)\Sigma\big(C(\cdot)[x(\cdot)]_{t_0}\big)(t), \\ P(t_0)(x(t_0) - x_0) = 0, \forall t \in \mathbb{T}_{t_0}, \end{cases} \tag{4.6}$$

where $[x(t)]_{t_0} = \begin{cases} 0 & \text{if } t \in [a,t_0) \\ x(t) & \text{if } t \in [t_0,\infty) \end{cases}$. The Cauchy problem (4.6) admits a mild solution if there exists an element $x(\cdot) \in L^{\text{loc}}_p(\mathbb{T}_{t_0};\mathbb{K}^n)$ such that for all $t \geq t_0$ we have

$$\begin{aligned} x(t) = \Phi(t,t_0)x_0 + \int_{t_0}^t \Phi(t,\sigma(s))P_\sigma(s)G^{-1}(s)B(s)\Sigma\big(C(\cdot)[x(\cdot)]_{t_0}\big)(s)\Delta s \\ + H(t)Q_\sigma(t)G^{-1}(t)B(t)\Sigma\big(C(\cdot)[x(\cdot)]_{t_0}\big)(t). \end{aligned} \tag{4.7}$$

Now, we define operators:

$$(\widehat{\mathbb{M}}_{t_0}u)(t) = \int_{t_0}^{t} \Phi(t,\sigma(s))P_\sigma(s)G^{-1}(s)B(s)u(s)\Delta s,$$
$$(\widetilde{\mathbb{M}}_{t_0}u)(t) = H(t)Q_\sigma(t)G^{-1}(t)B(t)u(t), (\mathbb{M}_{t_0}u)(t) = (\widehat{\mathbb{M}}_{t_0}u)(t) + (\widetilde{\mathbb{M}}_{t_0}u)(t).$$

$\mathbb{M}_{t_0}, \widehat{\mathbb{M}}_{t_0} \in \mathcal{L}(L_p([t_0,\infty);\mathbb{K}^m), L_p([t_0,\infty);\mathbb{K}^n))$ and there exists a constant $K_0 \geq 0$ such that $\|(\mathbb{M}_{t_0}u)(t)\| \leq K_0 \|u\|_{L_p([t_0,t];\mathbb{K}^m)}$, $t \geq t_0 \geq a$, $u|_{[t_0,t]} \in L_p([t_0,t];\mathbb{K}^m)$. Denote by $x(t;t_0,x_0)$ the (mild) solution of Cauchy problem (4.6). Then the formula (4.7) can be rewritten in form

$$x(t;t_0,x_0) = \Phi(t,t_0)x_0 + \big(\mathbb{M}_{t_0}\Sigma(C(\cdot)[x(\cdot;t_0,x_0)]_{t_0})\big)(t).$$

Theorem 4.4. *If Eq.* (4.6) *is of* index-1, *then it admits an unique mild solution $x(\cdot)$ with $P(\cdot)x(\cdot)$ to be absolutely continuous with respect to Δ-measure. Furthermore, for an arbitrary number $T > t_0$, there exist the positive constants $M_1 = M_1(T), M_2 = M_2(T)$ such that*

$$\|P(t)x(t)\| \leq M_1 \|P(t_0)x_0\|, \quad \|x(t)\|_{L_p([t_0,t];\mathbb{K}^n)} \leq M_2\|P(t_0)x_0\|, \quad \forall t \in [t_0,T].$$

Remark 4.5. Let the operator $\Sigma \in \mathcal{L}(L_p(\mathbb{T}_a;\mathbb{K}^q), L_p(\mathbb{T}_a;\mathbb{K}^m))$ be causal for all $t > a$ and $h \in L_p([a,t];\mathbb{K}^q)$. Then, by applying Theorem 4.4, we see that the function g, defined by $g(s) := P(t)x(t;\sigma(s),h(s))$, $s \in [a,t]$, belongs to $L_p([a,t];\mathbb{K}^n)$. Furthermore, set $y(t) := \int_s^t g(\tau)\Delta\tau$ then, by Theorem 1.27, we have $y^\Delta(t) = P_\sigma(t)h(t) + (\mathbb{W}y)(t)$, where $\mathbb{W}u := (P^\Delta + P_\sigma G^{-1}\bar{A})u + P_\sigma G^{-1}B\Sigma C(I+\mathbb{D})[u]_{t_0}$

4.2 Stability Radius under Dynamic Perturbations

Let Assumptions 4.1, 4.2 hold. The trivial solution of Eq. (4.5) is said to be *globally L_p-stable* if there exist the positive constants M_3, M_4 such that for all $t \geq t_0$, $x_0 \in \mathbb{K}^n$

$$\begin{aligned} &\|P(t)x(t;t_0,x_0)\|_{\mathbb{K}^n} \leq M_3 \|P(t_0)x_0\|_{\mathbb{K}^n}, \\ &\|x(t;t_0,x_0)\|_{L_p(T_{t_0};\mathbb{K}^n)} \leq M_4 \|P(t_0)x_0\|_{\mathbb{K}^n}. \end{aligned} \tag{4.8}$$

Definition 4.6. Let Assumptions 4.1, 4.2 hold. The complex (real) structured stability radius of Eq. (4.2) subject to linear, dynamic and causal perturbations in Eq. (4.5) is defined by $r_{\mathbb{K}}(E_\sigma, A; B, C; \mathbb{T}) = \inf \left\{ \begin{array}{l} \|\Sigma\|, \text{ the trivial solution of (4.5) is not} \\ \text{globally } L_p\text{-stable or (4.5) is not of index-1} \end{array} \right\}$.

For every $t_0 \in \mathbb{T}_a$, we define the following operators $\widehat{\mathbb{L}}_{t_0}u := C(\cdot)\widehat{\mathbb{M}}_{t_0}u$, $\widetilde{\mathbb{L}}_{t_0}u := C(\cdot)\widetilde{\mathbb{M}}_{t_0}u$, and $\mathbb{L}_{t_0}u := C(\cdot)\mathbb{M}_{t_0}u$. The operator $\mathbb{L}_{t_0}$ is called a input-output operator associated with the perturbed equation (4.5). It is clear that $\mathbb{L}_{t_0}, \widehat{\mathbb{L}}_{t_0} \in \mathcal{L}\big(L_p(\mathbb{T}_{t_0};\mathbb{K}^m), L_p(\mathbb{T}_{t_0};\mathbb{K}^q)\big)$ and $\|\mathbb{L}_{t_0}\|$, $\|\widehat{\mathbb{L}}_{t_0}\|$ are decreasing in t_0. Furthermore,

$$\|\widetilde{\mathbb{L}}_{t_0}\| = \Delta\text{-}\operatorname{esssup}_{t \geq t_0} \left\|CHQ_\sigma G^{-1}B\right\| \leq \|\mathbb{L}_{t_0}\|.$$

Since $\|\mathbb{L}_t\|$ is decreasing in t, there exists the limit $\|\mathbb{L}_\infty\| := \lim_{t\to\infty} \|\mathbb{L}_t\|$. Denote

$$\beta := \|\mathbb{L}_\infty\|^{-1}, \quad \gamma := \|\widetilde{\mathbb{L}}_a\|^{-1}, \text{ with the convention } \frac{1}{0} = \infty. \tag{4.9}$$

Lemma 4.8. *Suppose that $\beta < \infty$ and $\alpha > \beta$, where β is defined in* (4.9)*. Then, there exist an operator $\Sigma \in \mathcal{L}\big(L_p(\mathbb{T}_a;\mathbb{K}^q), L_p(\mathbb{T}_a;\mathbb{K}^m)\big)$, the functions $\tilde{y}, \tilde{z} \in L_p^{\text{loc}}(\mathbb{T}_a;\mathbb{K}^q)$ and a natural number $N_0 > 0$ such that*

i) $\|\Sigma\| < \alpha$*, Σ is causal and has a finite memory;*

ii) $\Sigma h(t) = 0$ *for every $t \in [0, N_0]$ and all $h \in L_p(\mathbb{T}_a;\mathbb{K}^q)$;*

iii) $\tilde{y} \in L_p^{\text{loc}}(\mathbb{T}_a;\mathbb{K}^q) \setminus L_p(\mathbb{T}_a;\mathbb{K}^q)$ *and* $\operatorname{supp} \tilde{z} \subset [0, N_0]$;

iv) $(I - \mathbb{L}_a\Sigma)\tilde{y} = \tilde{z}$.

Theorem 4.9. *Let Assumptions* 4.1, 4.2 *hold. Then*

$$r_{\mathbb{K}}(E_\sigma, A; B, C; \mathbb{T}) = \min\{\beta, \gamma\}, \tag{4.10}$$

where β, γ are defined in (4.9).

Remark 4.10. In case $\mathbb{T} = \mathbb{R}$, the formula (4.10) gives a formula for the stability radius in Du & Linh (2006), and in case $\mathbb{T} = \mathbb{Z}$ we obtain the radius of stability formula in Rodjanadid et al. (2009).

Remark 4.11. In case $\mathbb{T} = \mathbb{R}$ and $E = I$, the formula (4.10) gives a formula of the stability radius in Jacob (1998).

Example 4.12. Consider Eq. (4.3) with $E = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $A(t) = \begin{bmatrix} p(t) & p(t) & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ on time scale $\mathbb{T} = \bigcup_{k=0}^{\infty} \{3k\} \bigcup_{k=0}^{\infty} [3k+1, 3k+2]$, where $p(t) = \begin{cases} -\frac{1}{2} \text{ if } t = 3k, \\ -\frac{1}{4} \text{ if } t \in [3k+1, 3k+2]. \end{cases}$
It is easy to compute that $P = \widetilde{P} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $H = I, G^{-1} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & -1 \end{bmatrix}$. Assume that structured matrices $B = C = I$ in the perturbed equation (4.5). Therefore, we get $\|\mathbb{L}_{t_0}\| = 8$, $\|\widetilde{\mathbb{L}}_{t_0}\| = 1$. By Theorem 4.9 we obtain $r_{\mathbb{K}}(E_\sigma, A; B, C; \mathbb{T}) = \frac{1}{8}$.

Let $\Sigma \in L_\infty(\mathbb{T}_{t_0};\mathbb{K}^{m\times q})$ be a linear, causal operator defined by $(\Sigma u)(t) = \Sigma(t)u(t)$. Moreover, we have $\|\Sigma\| = \operatorname{esssup}_{t_0 \le t \le \infty} \|\Sigma(t)\|$.

Corollary 4.13 *Let Assumptions* 4.1, 4.2 *hold. Then, if $r_{\mathbb{K}}(E_\sigma, A; B, C; \mathbb{T}) > \|\Sigma\|$ then the perturbed equation* (4.5) *is globally L_p-stable.*

Remark 4.14. In case $\mathbb{T} = \mathbb{R}$ and $E = I$ and $\Sigma(\cdot) \in L_\infty(\mathbb{R}_{t_0};\mathbb{K}^{m\times q})$, the above corollary implies a lower bound for the stability radius in Hinrichsen et al. (1989).

Remark 4.15. By the Fourier-Plancherel transformation technique as in Hinrichsen & Pritchard (1986b) and Marks II et.al. (2008), if E, A, B, C are constant matrices and $p = 2$ then we can prove the equality $\|\mathbb{L}_{t_0}\| = \sup_{\lambda\in\partial S} \|C(A-\lambda E)^{-1}B\|$, where S is the domain of uniform exponential stability of the time scale $\mathbb{T}$,

$$S := \{\lambda \in \mathbb{C} : x^\Delta = \lambda x \text{ is uniformly exponentially stable}\}.$$

Moreover, $\|\widetilde{\mathbb{L}}_{t_0}\| = \lim_{\lambda\to\infty} \|C(A-\lambda E)^{-1}B\|$. Thus, we obtain the stability radius formula in Du et al. (2011): $r(E, A; B, C; \mathbb{T}) = \dfrac{1}{\sup_{\lambda\in\partial S\cup\infty} \|C(A-\lambda E)^{-1}B\|}$.

4.3 Stability Radius under Structured Perturbations on Both Sides

Now, in this section, we consider Eq. (4.2) subject to perturbations acting both derivative and right-hand side of the form

$$(E_\sigma + B_{1\sigma}\Sigma_{1\sigma}C_{1\sigma})(t)x^\Delta(t) = (A + B_2\Sigma_2C_2)(t)x(t), \quad t \ge t_0. \tag{4.11}$$

where $B_i \in L_\infty(\mathbb{T}_{t_0}; \mathbb{K}^{n\times m}), C_i \in L_\infty(\mathbb{T}_{t_0}; \mathbb{K}^{q\times n})$ are given matrices, Σ_i are perturbations in $L_\infty(\mathbb{T}_{t_0}; \mathbb{K}^{m\times q})$, for only $i = 1, 2$. We define the set of admissible perturbations $\mathbb{S} = \mathbb{S}(E; B_1, C_1) := \{(\Sigma_1, \Sigma_2) | \ker(E + B_1\Sigma_1C_1) = \ker(E)\}$.

Lemma 4.16. *The following assertions hold*

i) $Q_\sigma Q^\Delta H Q_\sigma = 0$; ii) $Q_\sigma Q^\Delta P = Q^\Delta$; iii) $I + Q^\Delta H Q_\sigma$ *is invertible*;

iv) $(I + Q^\Delta H Q_\sigma)G^{-1} = (E_\sigma - AHQ_\sigma)^{-1}$, $Q_\sigma G^{-1} = Q_\sigma(E_\sigma - AHQ_\sigma)^{-1}$.

Define $\bar{\mathbb{A}} = \mathbb{A} - \mathbb{E}_\sigma Q^\Delta$, $\mathbb{G} := \mathbb{E}_\sigma - \bar{\mathbb{A}} H Q_\sigma$ and $B := \begin{bmatrix} B_{1\sigma} & B_2 \end{bmatrix}$, $\Sigma_b := \begin{bmatrix} \Sigma_{1\sigma} & 0 \\ 0 & \Sigma_2 \end{bmatrix}$.

Lemma 4.17. *Assume that Eq.* (4.2) *is of* index-1. *If* $(\Sigma_1, \Sigma_2) \in \mathbb{S}$ *such that* $\|\Sigma_b\| < \frac{1}{\|FB\|}$, *then the perturbed equation* (4.11) *is also of* index-1.

Lemma 4.18. *Let Eq.* (4.2) *be of* index-1. *Then Eq.* (4.5) *is equivalent to Eq.* (4.11) *with the perturbation* $\Sigma = (I + \Sigma_b FB)^{-1}\Sigma_b$.

Definition 4.19. Let Assumptions 4.1, 4.2 hold. The complex (real) structured stability radius of Eq. (4.2) subject to linear structured perturbations in Eq. (4.11) is defined by

$$r_{\mathbb{K}}(E_\sigma, A; B_1, C_1, B_2, C_2; \mathbb{T}) = \inf \left\{ \begin{matrix} \|\Sigma_b\|, \text{ the trivial solution of (4.11) is not} \\ \text{globally } L_p\text{-stable or (4.11) is not of index-1} \end{matrix} \right\}.$$

Theorem 4.20. *Let Assumptions* 4.1, 4.2 *hold, and* β, γ *are defined in* (4.9). *The complex (real) structured stability radius of Eq.* (4.2) *subject to linear structured perturbations in Eq.* (4.11) *satisfies*

$$r_{\mathbb{K}}(E_\sigma, A; B_1, C_1, B_2, C_2; \mathbb{T}) \ge \begin{cases} \frac{\min\{\beta;\gamma\}}{1+\|FB\|\min\{\beta;\gamma\}} & \text{if } \beta < \infty \text{ or } \gamma < \infty, \\ \frac{1}{\|FB\|} & \text{if } \beta = \infty \text{ and } \gamma = \infty. \end{cases}$$

Example 4.21. Consider the IDE, $Ex^{\Delta} = Ax$, $E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $A = \begin{bmatrix} -1 & \frac{1}{2} & 0 \\ \frac{1}{2} & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix}$. Assume that this equation is subject to structured perturbations $E \rightsquigarrow \mathbb{E}$, $A \rightsquigarrow \mathbb{A}$

$$\mathbb{E} = \begin{bmatrix} 1+\delta_1(t) & \delta_1(t) & \delta_1(t) \\ \delta_1(t) & 1+\delta_1(t) & \delta_1(t) \\ 0 & 0 & 0 \end{bmatrix}, \mathbb{A} = \begin{bmatrix} -1 & \frac{1}{2} & 0 \\ \frac{1}{2}+\delta_2(t) & -1+\delta_2(t) & 1+\delta_2(t) \\ \delta_2(t) & \delta_2(t) & -1+\delta_2(t) \end{bmatrix},$$

where $\delta_i(t), i = 1,2$, are perturbations. It is easy to see that this model can be rewritten in form (4.11) with $B_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $B_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$, $C_1 = C_2 = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$. In this example, we have $P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $Q = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. By simple computations, we get $B = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}$, $F = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$, $C = \begin{bmatrix} -\frac{1}{2} & -\frac{1}{2} & 0 \\ -1 & -1 & 0 \end{bmatrix}$. Therefore $\|FB\| = 3$ and $C(A - \lambda E)^{-1}B = \dfrac{1}{(\lambda+1)^2 - \frac{1}{4}} \begin{bmatrix} \lambda + \frac{3}{2} & \lambda + \frac{3}{2} \\ 2\lambda + 3 & 2\lambda + 3 \end{bmatrix}$. Let $\mathbb{T} = \bigcup_{k=1}^{\infty}[2k, 2k+1]$. Then, the domain of uniformly exponential stability $S = \{\lambda \in \mathbb{C} : \Re\lambda + \ln|1+\lambda| < 1\}$. Using Remark 4.15, we yield $\beta = \frac{1}{8}, \gamma = +\infty$. Thus, by applying Theorem 4.20, we obtain

$$r_{\mathbb{K}}(E_\sigma, A; B_1, C_1, B_2, C_2; \mathbb{T}) \geq \frac{1}{11}.$$

Corollary 4.22. *Let Assumptions* 4.1, 4.2 *hold. The complex* (*real*) *structured stability radius of Eq.* (4.2) *subject to linear unstructured perturbations* $E \rightsquigarrow E + \Sigma_1, A \rightsquigarrow A + \Sigma_2$ *satisfies*

$$r_{\mathbb{K}}(E_\sigma, A; I; \mathbb{T}) \geq \begin{cases} \frac{\min\{l(E,A), \|HQ_\sigma G^{-1}\|_\infty^{-1}\}}{k_1 + k_2 \min\{l(E,A), \|HQ_\sigma G^{-1}\|_\infty^{-1}\}} & \text{if } Q \neq 0 \text{ or } l(E,A) < \infty, \\ \frac{1}{k_2} & \text{if } Q = 0 \text{ and } l(E,A) = \infty. \end{cases}$$

with the convention $\|HQ_\sigma G^{-1}\|_\infty^{-1} = \infty$ *if* $\|HQ_\sigma G^{-1}\|_\infty = 0$.

Remark 4.23. In case $\mathbb{T} = \mathbb{R}$, this corollary is a result concerning the lower bound of the stability radius in Berger (2014).

Example 4.24. Consider Eq. (4.3) with $E, A, \mathbb{T}$ in Example 4.12. Then, we can compute. It is not difficult to imply $\|p\|_\infty = \dfrac{1}{2}$, $k_1 = k_2 = 1$. Hence, by Corollary 4.22, we obtain

$$r_{\mathbb{K}}(E_\sigma, A; I; \mathbb{T}) \geq \frac{\frac{1}{8}}{1+\frac{1}{8}} = \frac{1}{9}.$$

CONCLUSIONS

The dissertation has achieved the following main results:

1. Introducing of the definition for Lyapunov exponent and using it to study the stability of linear dynamic equations on time scales.

2. Establishing the robust stability of implicit dynamic equations with Lipschitz perturbations, and extending Bohl-Perron type stability theorem for implicit dynamic equations on time scales.

3. Suggesting the concept for Bohl exponent on time scales and studying the relation between exponential stability and the Bohl exponent when dynamic equations under perturbations acting on the system coefficients.

4. Recommending the radius of stability formula for implicit dynamic equations on time scales under some structured perturbations acting on the right-hand side or both side-hands.

LIST OF THE AUTHOR'S SCIENTIFIC WORKS

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2. Thuan D.D., Nguyen K.C., Ha N.T., and Du N.H. (2019), Robust stability of linear time-varying implicit dynamic equations: A general consideration, *Mathematics of Control, Signals, and Systems*, **31(3)**, 385–413 (SCI).

3. Thuan D.D., Nguyen K.C., Ha N.T., and Quoc P.V. (2020), On stability, Bohl exponent and Bohl-Perron theorem for implicit dynamic equations, *International Journal of Control*, Published online, (SCI).