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STABILITY AND ROBUST STABILITY OF LINEAR DYNAMIC EQUATIONS ON TIME SCALES

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DECLARATION

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CONTENTS

ABTRACT

The characterization of analysis on time scales is the unification and generalization of results obtained on the discrete and continuous-time analysis. For the last decades, the studies of analysis on time scales have led to many more general results and had many applications in different fields. One of the most important problems in this research field is to study the stability and robust stability of dynamic equations on time scales. The main content of the dissertation will present our new results obtained about this subject. The dissertation is divided into four chapters.

Chapter 1 presents the background knowledge on a time scale in preparation for upcoming results in the next chapters.

In Chapter 2, we introduce the concept of Lyapunov exponents for functions defined on time scales and study some of their basic properties. We also establish the relation between Lyapunov exponents and the stability of a linear dynamic equation $x^\Delta = A(t)x$. This does not only unify but also extend well-known results about Lyapunov exponents for continuous and discrete systems.

Chapter 3 develops the stability theory for IDEs $E_\sigma(t)x^\Delta = A(t)x$. We derive some results about the robust stability of these equations subject to Lipschitz perturbations, and the so-called Bohl-Perron type stability theorems are extended for IDEs. Finally, the notion of Bohl exponents is introduced and characterized the relation with exponential stability. Then, the robustness of Bohl exponents of equations subject to perturbations acting on the system data is investigated.

In Chapter 4, the robust stability for linear time-varying IDEs $E_\sigma(t)x^\Delta = A(t)x + f(t)$ is studied. We consider the effects of uncertain structured perturbations on all system's coefficients. A stability radius formula with respect to dynamic structured perturbations acting on the right-hand side is obtained. When structured perturbations affect both the derivative and right-hand side, we get lower bounds for stability radius.

LIST OF NOTATIONS

$\mathbb{T}$	time scale
$\mathbb{T}^\kappa$	$\mathbb{T} \setminus \{T_{\max}\}$ if $\mathbb{T}$ has a left-scattered maximum $T_{\max}$
$\mathbb{T}_\tau$	$\{t \in \mathbb{T} :\ t \geq \tau\}$, for all $\tau \in \mathbb{T}$
$\sigma(\cdot)$	forward jump operator
$\varrho(\cdot)$	backward jump operator
$\mu(\cdot)$	graininess function
$f^\Delta(\cdot)$	derivative of function f on time scales
$e_\alpha(t,s)$	exponential function with a parameter α on time scales
Log	principal logarithm function with the valued-domain is $[-i\pi, i\pi)$
$\kappa_L[f]$	Lyapunov exponent of a function $f(\cdot)$ on time scales
$\kappa_{\mathcal{B}}(E,A)$	Bohl exponent of an equation $E(t)x^\Delta = A(t)x$ on time scales
$\mathbb{N}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$	sets of natural, rational, real, complex numbers
$\mathbb{N}_0$	$\mathbb{N} \cup \{0\}$
$\mathbb{R}^+$	set of positive real numbers
$\mathbb{K}$	a field, to be replaced by set $\mathbb{R}$ or $\mathbb{C}$, respectively
$\mathbb{K}^{m\times n}$	linear space of $m \times n$-matrices on $\mathbb{K}$
$\mathrm{C}(X,Y)$	space of continuous functions from X to Y
$\mathrm{C}^1(X,Y)$	space of continuously differentiable functions from X to Y
$\mathrm{C}_{\mathrm{rd}}(\mathbb{T},X)$	space of rd-continuous functions $f : \mathbb{T} \to X$
$\mathrm{C}^1_{\mathrm{rd}}(\mathbb{T},X)$	space of rd-continuously differentiable functions $f : \mathbb{T}^\kappa \to X$
$\mathcal{R}(\mathbb{T},X)$	set of regressive functions $f : \mathbb{T} \to X$
$\mathcal{R}^+(\mathbb{T},X)$	set of positive regressive functions $f : \mathbb{T} \to X$
$\mathrm{C}_{\mathrm{rd}}\mathcal{R}(\mathbb{T},X)$	space of rd-continuous and regressive functions $f : \mathbb{T} \to X$
$\mathrm{PC}(X,\mathbb{K}^{m\times n})$	set of piecewise continuous matrix functions $D : X \to \mathbb{K}^{m\times n}$

$\mathrm{PC_b}(X, \mathbb{K}^{m\times n})$	set of bounded, piecewise continuous matrix functions $D : X \to \mathbb{K}^{m\times n}$
$\mathrm{Gl}(\mathbb{R}^m)$	set of linear automorphisms of $\mathbb{R}^m$
$\Im\lambda$	imaginary part of a complex number λ
$\Re\lambda$	real part of a complex number λ
$\operatorname{im} A$	image of an operator A
$\ker A$	kernel of an operator A
$\operatorname{rank} A$	rank of a matrix A
$\det A$	determinant of a matrix A
$\operatorname{trace} A$	trace of a matrix A
$\sigma(A)$	set of eigenvalues of a matrix A
$\sigma(A, B)$	set of complex solutions to an equation $\det(\lambda A - B) = 0$
$\sup F,\ \inf F$	supremum, infimum of a function F
$\operatorname{esssup} F$	essential supremum of a function F
$\operatorname{supp} F$	support of a function F
DAE	differential-algebraic equation
IDE	implicit dynamic equation
ODE	ordinary differential equation
IVP	initial value problem

INTRODUCTION

Continuous and discrete-time dynamic systems as a whole (hybrid systems) are of undoubted interest in many applications. The mathematical analysis developed on time scales allows us to consider real-world phenomena in a more accurate description/modeling. The time scale calculus has tremendous potential for applications or practical problems. For example, dynamic equations on time scales can model insect populations that evolve continuously while in season (and may follow a difference scheme with the variable step-size), die out in (say) winter, while their eggs are being incubated or dormant, and then hatch in a new season, giving rise to a non-overlapping population.

The analysis on time scales was introduced in 1988 by Stefan Hilger in his Ph.D. dissertation (supervised by Prof. Bernd Aulbach, 1947-2005) [35]. We may say that the theory of analysis on time scales is established in order to build bridges between continuous and discrete-time systems and unify two these ones. Further, studying the theory of time scales has led to many important applications, e.g., in the study of insect population models, neural networks, heat transfers, quantum mechanics, and epidemic models... As soon as this theory was born, it has attracted the attention of many mathematical researchers. There have been a lot of works on the theory of time scales published over the years, see monographs [9, 10, 60]. Many familiar results on not only qualitative but also quantitative theory in continuous and discrete-time were "shifted" and "generalized" to the case of time scales, such as stability theory, oscillation, boundary value problem...

One of the most important problems in the analysis on time scales is to investigate dynamic equations. Many results concerning differential equations are carried over quite easily to the corresponding results for difference equations, while the others seem to have complete differences in nature from their continuous counterparts. The investigation of dynamic equations on time scales reveals such discrepancies between the differential and difference equations. Moreover, it helps us avoid proving twice a result, one for differential equations and one for difference equations. The general idea is to prove results for dynamic equations where the domain of a function is called a time scale and denoted by $\mathbb{T}$, which is an arbitrary, nonempty, closed subset of real numbers $\mathbb{R}$. It is also called a "measure chain". In case $\mathbb{T} = \mathbb{R}$, the general results yield the ones concerning ordinary differential equations. In case $\mathbb{T} = \mathbb{Z}$, the same general results yield the ones for difference equations. However, since there are many more complex time scales than $\mathbb{R}$ and $\mathbb{Z}$, investigating the theory of dynamic equations on time scales leads to other general results. Especially, there are still many open problems in studying dynamic equations on time scales. That is why so far the analysis on time scales has still been an attractive topic in mathematical analysis.

The aim of this dissertation is to use some basic concepts, such as the Bohl exponent, Lyapunov exponent, and stability radius to investigate the stability, robust stability of dynamic equations on time scales.

We know, that the Bohl exponent and Lyapunov exponent are used to investigate the solution's asymptotic behavior of differential equations. The Lyapunov exponent was introduced by A.M. Lyapunov (1857-1918) in his Ph.D. Dissertation in 1892. The Bohl exponent was declared by P. Bohl (1865-1921) in 1913 in his article[1]. Both of them are used to describe the exponential growth of solutions of the equation $\dot{x} = A(t)x$.

The Bohl exponent has been successfully used to characterize exponential stability and to derive robustness results for ODEs, see, e.g. [18, 39]. In 2008, Chyan et al. [14] generalized several ODE results concerning the Bohl exponent to DAEs which the leading term is supposed to be singular. Note that the authors treated only linear DAEs of index-1. Next, in 2009, Linh and Mehrmann [50] investigated Bohl spectral intervals and Bohl exponent of particular solutions and fundamental solution matrices of DAEs. However, the Bohl exponent of the system does not lie in the focus of both these works. Both [14] and [50] avoid the problem of a proper definition of Bohl exponents for general implicit differential equations. In Berger's articles [6, 7] (2012, 2014), he developed the theory of Bohl exponent for linear time-varying DAEs. Results of these articles are generalizations of ODE results in [18, 39] and the others to DAEs. Recently, in 2016, Du et al. [26] introduced the notation of Bohl exponents and characterize the relation between the exponential stability and Bohl exponent of linear singular systems of

[1] Bohl P. (1913), Uber Differentialungleichungen, J.F.d. Reine Und Angew. Math., **144**, 284–133.

difference equations with variable coefficients, in which, the robustness of Bohl exponents with respect to allowable two-side perturbations was also investigated.

The Lyapunov exponents were first introduced in 1892 under the name of *characteristic numbers* for finite-dimensional differential equations. As far as we know, the authors have used only the second Lyapunov method (or the method of Lyapunov functions) to investigate whether the dynamic equation, $x^{\Delta} = A(t)x$, is stable or not. This method is simpler than the approach on time scales, see [15, 30, 44, 60]. Meanwhile, the first Lyapunov method (the method of Lyapunov exponents) is a quite classical and basic concepts for studying differential and difference equations [53, 60], and it is a strong tool to study the stability of linear systems. So far, there have been no works dealing with the concept of Lyapunov exponents and the stability for functions defined on time scales. The main reason for this situation is that the traditional approach to Lyapunov exponents via logarithm functions is no longer valid because there is no reasonable definition for logarithm functions on time scales, which one regards as the inverse of the exponential function $e_{p(t)}(t,s)$. There have been some works trying to approach this notion, e.g., in [8] (2005), M. Bohner have proposed two approaches. The first: compares solutions of the Euler-Cauchy dynamic equation $t\sigma(t)x^{\Delta\Delta} - 3tx^{\Delta} + 4x = 0$ on time scale $\mathbb{T}$, and the differential equation $t^2\ddot{x} - 3t\dot{x} + 4x = 0$ on $\mathbb{R}$, the logarithm function on time scale is derived as follows $L_p(t,t_0) = \int_{t_0}^{t} \frac{\Delta\tau}{\tau + 2\mu(\tau)}$. In the second approach, we define a logarithm function by $L_p(t,t_0) = \int_{t_0}^{t} \frac{p^{\Delta}(\tau)\Delta\tau}{p(\tau)}$. However, both definitions are not good enough since $L_{pq}(t,t_0) \neq L_p(t,t_0) + L_q(t,t_0)$.

DAEs are mathematical models arising in various applications, such as multibody mechanics, electrical circuits, chemical engineering, etc., see [11, 46, 48]. Similarly, implicit difference equations also occur in different fields, such as population dynamics, economics, systems, and control theory, etc., see [51, 54, 55]. Therefore, it is meaningful to combine these equations by the theory of singular dynamic systems on an arbitrary time scale. This theory has been found promising because it demonstrates the interplay between the theory of continuous and discrete-time systems, see, e.g. [2, 5, 17, 36], and also allows to analyze the stability of dynamical systems on nonuniform time domains which are the subsets of $\mathbb{R}$. Then, the time-varying IDE

$$E_{\sigma}(t)x^{\Delta}(t) = A(t)x(t) + f(t), t \geq t_0, \tag{1}$$

can be considered as a unified and generalized form of time-varying DAE

$$E(t)\dot{x}(t) = A(t)x(t) + f(t), t \geq t_0,$$

and time-varying implicit difference equation

$$E_{n+1}x(n+1) = A_n x(n) + f_n, n \geq n_0.$$

Thus, it plays an important role in mathematical modeling with many applications. However, the singularity of the leading coefficient introduces many difficulties for the analysis of IDE (1), for example the explicit computation of solutions is impossible at the first observation. Even the solvability of the initial value problem is doubtful. Since the dynamics of Equation (1) are constrained and combined between differential and difference components, some extra difficulties appear in the stability analysis and also in the numerical analysis of IDEs are characterized by index concepts, see [12, 33, 46, 48].

On the other hand, there have been extensive works on studying of robust measures, where one of the most powerful ideas is the concept of stability radius, introduced by Hinrichsen and Pritchard [37, 38]. The so-called stability radius is defined as the norm of the smallest perturbations destabilizing the equation, and it answers the question of how robust is a stability property of a system when the system comes under the effect of uncertain perturbations. The analysis of robust stability is a subject that has recently attracted an attention from researchers, and there have been many published scientific works, see [21, 25, 41, 42, 67, 68]. There are many results for the stability radius of the time-invariant linear systems, see [20, 27, 28, 43, 61, 63]. However, results for the stability radius of time-varying systems are not many. For ordinary differential equations $\dot{x} = A(t)x, t \geq 0$, subject to structured perturbations of the form $\dot{x} = A(t)x + B(t)\Sigma(C(\cdot)x(\cdot))(t)$, $t \geq 0$, the notation of a complex stability radius of linear time-varying systems is defined by Hinrichsen et al. (1992) [40]

$$r_{\mathbb{C}}(I, A; B, C) := \inf \left\{ \begin{array}{l} \|\Sigma\|_{L_\infty}, \Sigma \in \mathrm{PC_b}(\mathbb{R}^+, \mathbb{C}^{m\times q}), \text{ and} \\ (I, A; B, C) \text{ is not exponential stable} \end{array} \right\},$$

where $\Sigma \in \mathrm{PC_b}(\mathbb{R}^+, \mathbb{C}^{m\times q})$ is an unknown, bounded, time-varying disturbance matrix, $B \in \mathrm{PC}(\mathbb{R}^+, \mathbb{C}^{n\times m})$, $C \in \mathrm{PC}(\mathbb{R}^+, \mathbb{C}^{q\times n})$ are given scaling-matrices defining the structure of perturbation. The first stability radius formula is derived by Jacob (1998) [45],

$$r_{\mathbb{K}}(I, A; B, C) = \sup_{t_0 \geq 0} \|\mathbb{L}_{t_0}\|^{-1},$$

where $\Sigma \in L_\infty([0,\infty),\mathbb{K}^{m\times q})$ is an unknown, causal, dynamical disturbance operator, $B \in L_\infty([0,\infty),\mathbb{K}^{n\times m})$, $C \in L_\infty([0,\infty),\mathbb{K}^{q\times n})$ are given scaling matrices defining the structure of perturbation, and $\mathbb{L}_{t_0}$ defined by

$$(\mathbb{L}_{t_0}u)(t) := C(t)\int_{t_0}^{t}\Phi(t,\tau)B(\tau)u(\tau)d\tau, t \geq t_0 \geq 0, u \in L_p([t_0,\infty);\mathbb{K}^m).$$

After that, in [24], Du and Linh (2006) has investigated the stability radius of linear time-varying DAEs of index-1

$$E(t)\dot{x} = A(t)x,\ t \geq 0, \tag{2}$$

with respect to only right-hand side perturbations,

$$E(t)\dot{x} = A(t)x + B(t)\Sigma(C(\cdot)x(\cdot))(t), t \geq 0,$$

where E is pointwise singular, $\Sigma \in L_\infty([0,\infty),\mathbb{K}^{m\times q})$ is an unknown disturbance operator supposed to be linear, dynamic and causal, and $B \in L_\infty([0,\infty),\mathbb{K}^{n\times m})$, $C \in L_\infty([0,\infty),\mathbb{K}^{q\times n})$ are given matrices defining the structure of perturbation. The stability radius formula has been derived

$$r_{\mathbb{K}}(E,A;B,C) = \min\left\{\sup_{t_0\geq 0}\|\mathbb{L}_{t_0}\|^{-1}, \|\widetilde{\mathbb{L}}_0\|^{-1}\right\},$$

where

$$(\mathbb{L}_{t_0}u)(t) := C(t)\int_{t_0}^{t}\Phi(t,\tau)PG^{-1}B(\tau)u(\tau)d\tau + CQG^{-1}B(t)u(t),$$
$$(\widetilde{\mathbb{L}}_{t_0}u)(t) := CQG^{-1}B(t)u(t),\ t \geq t_0 \geq 0, u \in L_p([t_0,\infty);\mathbb{K}^m).$$

In 2009, Rodjanadid et al. [69] studied the stability radius of linear implicit difference equation varying in time with index-1, $E_n x(n+1) = A_n x(n)$, $n \in \mathbb{N}$, subject to structured parameter perturbations of the form: $E_n x(n+1) = A_n x(n) + B_n\Sigma(C_{\cdot}x(\cdot))(n)$, $n \in \mathbb{N}$, where $B \in \mathbb{K}^{d\times m}, C \in \mathbb{K}^{q\times d}$, and $\Sigma \in \mathcal{L}(L_p([0,\infty);\mathbb{K}^q), L_p([0,\infty);\mathbb{K}^m)$ is the causal perturbation operator. The stability radius formula is obtained

$$r_{\mathbb{K}}(E,A;B,C) = \min\left\{\sup_{n_0\geq 0}\|\mathbb{L}_{n_0}\|^{-1}, \|\widetilde{\mathbb{L}}_0\|^{-1}\right\},$$

where

$$(\mathbb{L}_{n_0}u)(n) := C_n\sum_{k=0}^{n-1}\Phi(n,k+1)P_kG_k^{-1}B_ku(k) + (\widetilde{\mathbb{L}}_0u)(n),$$

$$(\widetilde{\mathbb{L}}_0 u)(n) := C_n T_n Q_n G_n^{-1} B_n u(n), \text{ and } (\mathbb{L}_{n_0} u)(n) = (\mathbb{L}_0 [u]_{n_0})(n).$$

Recently, Berger (2014) [7] also investigates the stability radius of Equation (2) under unstructured perturbation acting on the left-hand side,

$$(E(t) + \Sigma(t))\dot{x} = A(t)x, \quad t \in \mathbb{R}^+,$$

where $E, A \in C(\mathbb{R}^+, \mathbb{R}^{n\times n})$, the perturbation $\Sigma \in C(\mathbb{R}^+, \mathbb{R}^{n\times n})$. It has got some lower bounds for the stability radius

$$r(E,A) \geq \begin{cases} \frac{\min\{l(E,A), \|QG^{-1}\|_\infty^{-1}\}}{\kappa_1 + \kappa_2 \min\{l(E,A), \|QG^{-1}\|_\infty^{-1}\}} & \text{if } Q \neq 0, \\ \frac{l(E,A)}{\kappa_1 + \kappa_2 l(E,A)}, & \text{if } Q = 0 \text{ and } l(E,A) < \infty, \\ \frac{1}{\kappa_2}, & \text{if } Q = 0 \text{ and } l(E,A) = \infty, \end{cases}$$

where

$$l(E,A) := \lim_{t_0 \to \infty} \|\mathbb{L}_{t_0}\|^{-1} = \sup_{t_0 \geq 0} \|\mathbb{L}_{t_0}\|^{-1},$$

$$(\mathbb{L}_{t_0} u)(t) := \int_{-t_0}^{t} \Phi(t,\tau) P G^{-1} u(\tau) d\tau + Q(t) G^{-1}(t) u(t).$$

Therefore, it is meaningful to continue studying the stability radius for time-varying IDEs subject to structured perturbations acting on both sides. However, there are some difficulties in solving this problem in which the operator of the left shift may not exist on an arbitrary time scale and structured perturbations acting on the coefficient of derivative can easily change the index and the stability of IDEs. It is worth remarking that if the perturbed system does not have the index-1 property, then the well-posedness of the initial value problem cannot be expected. Hence, it is quite natural to require the index-1 property for the perturbed system.

The dissertation is mainly based on the obtained results in three articles and it is divided into four chapters, apart from the introduction and conclusion parts. Chapter 1 presents the basic knowledge of analysis on time scales in order to prepare for investigations in Chapter 2, Chapter 3, and Chapter 4.

In Chapter 2, we introduce an approach to the first Lyapunov method for the dynamic equations of the form

$$x^\Delta = A(t)x, \quad t \in \mathbb{T} \tag{3}$$

on time scales, where A is a regressive, rd-continuous $n \times n$-matrix function. Firstly, in Section 2.1, we define the Lyapunov exponent for a function

$f : \mathbb{T}_a \to \mathbb{K}$ on time scale and prove its fundamental properties. Although we cannot define the logarithm function on time scales, the idea of comparing the growth rate of a function with exponential functions in the definition of Lyapunov exponent is still useful on the time scales. Therefore, instead of considering the limit $\limsup_{t\to\infty} \frac{1}{t} \ln |f(t)|$, we can use the oscillation of the ratio $\frac{|f(t)|}{e_\alpha(t,t_0)}$ as $t \to \infty$ in the parameter α to define Lyapunov exponent of the function f on time scale $\mathbb{T}$. Especially, in this section, we get a necessary and sufficient condition for the existence of normal Lyapunov exponent in Lemma 2.2. Next, in Section 3.2, we investigate the Lyapunov exponent of solutions of Equation (3), with the initial value $x(t_0) = x_0$. We have also obtained an estimate for Lyapunov exponent of a nontrivial solution of Equation (3) in Theorem 2.15, and for Lyapunov inequality in Theorem 2.19. Finally, in Section 2.3, we study the relation between Lyapunov spectrum and the stability of Equation (3) with the bounded condition of matrix A. If A is a time-varying matrix, we obtain necessary and sufficient conditions such that Equation (3) is exponentially asymptotically stable, and uniformly exponentially asymptotically stable in Theorems 2.24, 2.25. If A is a constant matrix, we also have a sufficient condition for the exponentially asymptotic stability of Equation (3) in Theorem 2.26.

Considering linear time-varying IDE (1) with index-1, $E_\sigma \in L_\infty^{\text{loc}}(\mathbb{T}; \mathbb{K}^{n\times n})$ and is supposed to be singular, $A \in L_\infty^{\text{loc}}(\mathbb{T}; \mathbb{K}^{n\times n})$, and $\ker A$ is absolutely continuous. The corresponding homogeneous equation is

$$E_\sigma(t)x^\Delta(t) = A(t)x(t), \quad t \in \mathbb{T}. \tag{4}$$

The main content of Chapter 3 is about the stability, Bohl exponent and relative results for IDEs. Firstly, in Section 3.1, we prove that the solution of linear time-varying IDE (1) is unique and has the form (3.10). Next, in Section 3.2, we affirm that if Equation (4) is exponentially stable, then under small Lipschitz perturbations, $f(\cdot) = F(\cdot, x(\cdot))$, it is still exponentially stable, Theorem 3.10. In addition, if all solutions of the initial value problem (1) are bounded, then the index-1 IDE (4) is exponentially stable and vice versa, Theorem 3.14. Finally, in Section 3.3, we introduce the definition of Bohl exponent for IDEs and discuss some properties, especially, the relation among exponential stability, Bohl exponent of Equation (4) and solutions of Cauchy problem (1) is derived, Theorem 3.23. We also get results about the robustness of Bohl exponent with respect to perturbations acting only on the right-hand side, Theorem 3.26, or on both sides, Theorems 3.27.

In Chapter 4, we investigate the general stability radius formula of IDE (4) subject to structured perturbations acting on the right-hand side or both sides. Firstly, in Section 4.1, we use results in Section 3.1, Chapter 3 to get the formula of solution (4.8) of Equation (4) subject to the dynamic perturbation of the form

$$E_{\sigma}(t)x^{\Delta}(t) = A(t)x(t) + B(t)\Sigma(C(\cdot)x(\cdot))(t), t \in \mathbb{T}.$$

Next, in Section 4.2, we derive a formula for the structured stability radius of Equation (4) with respect to dynamic perturbations acting on the right-hand side, Theorem 4.9. This result extends many previous results for the stability radius of time-varying differential and difference equations, time-varying differential-algebraic equations and implicit difference equations in [24, 39, 45, 69]. Finally, in Section 4.3, we study the stability radius involving structured perturbations acting on both sides and obtain some lower bounds in Theorem 4.20 and Corollary 4.22. In case $\mathbb{T} = \mathbb{R}$ and the equation is subjected to unstructured perturbations, we obtain a lower bound for the stability radius in [7].

The dissertation was completed at Hanoi Pedagogical University 2, Course 2015 - 2019 and presented at the seminar of the Faculty of Mathematics, HPU2. The main results were also reported at

1. Vietnam - Korea Joint Workshop on Dynamical Systems and Related Topics (Vietnam Institute for Advanced Study in Mathematics, Hanoi, Vietnam, March 02-05, 2016);
2. the 2nd Pan-Pacific International Conference on Topology and Applications (Pusan National University, Busan, Korea, November 13-17, 2017);
3. the 9th Vietnam Mathematical Congress (Vietnamese Mathematical Society, Nhatrang, Vietnam, August 14-18, 2018); and
4. International Conference on Differential Equations and Dynamical Systems (Hanoi Pedagogical University 2 and Institue of Mathematics - Vietnam Academy of Science and Technology, Vinhphuc, September 05-07, 2019).

CHAPTER 1

PRELIMINARIES

In this chapter, we introduce some basic concepts about analysis theory on time scales to study the stability and robust stability of dynamic equations. In 1988, the theory of analysis on time scales was introduced by Stefan Hilger [35] in his Ph.D. dissertation in order to unify and extend continuous and discrete calculus. The content in Chapter 1 is referenced to [9], [10] and the material therein.

1.1 Time Scale and Calculations

1.1.1 Definition and Example

Definition 1.1 ([9], page 1)**.** The *time scale* denoted by $\mathbb{T}$ is an arbitrary, nonempty, and closed subset of the set of real numbers $\mathbb{R}$.

For example, the set of real numbers $\mathbb{R}$, integers $\mathbb{Z}$, natural numbers $\mathbb{N}$, and nonegative integers $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $[a, b]$ are time scales. The set of rational numbers $\mathbb{Q}$, irrational numbers $\mathbb{R} \backslash \mathbb{Q}$, complex numbers $\mathbb{C}$, and the open interval $(0, 1)$ are not time scales. We assume throughout this dissertation that the time scale $\mathbb{T}$ has an inherited topology of $\mathbb{R}$.

Definition 1.2 ([9], page 1)**.** Let $\mathbb{T}$ be a time scale. For all $t \in \mathbb{T}$, we define

i) the *forward operator* $\sigma : \mathbb{T} \to \mathbb{T}$ by $\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}$;

ii) the *backward operator* $\varrho : \mathbb{T} \to \mathbb{T}$ by $\varrho(t) := \sup\{s \in \mathbb{T} : s < t\}$, and

iii) the *graininess* function $\mu : \mathbb{T} \to [0, \infty)$ by $\mu(t) := \sigma(t) - t$.

In addition, we set $\sigma(M) = M$ if $\mathbb{T}$ has a maximum value M, and $\varrho(m) = m$ if $\mathbb{T}$ has a minimum value m.

Definition 1.3 ([10], page 10)**.** For all $x, y \in \mathbb{T}$, we have:

i) the *circle plus*, $\oplus$: $x \oplus y := x + y + \mu(t)xy$;

ii) the *symmetric element* $\ominus x$ of x: $\ominus x := \dfrac{-x}{1+\mu(t)x}$, and

iii) the *circle minus*, $\ominus$: $x \ominus y := \dfrac{x-y}{1+\mu(t)y}$.

Definition 1.4 ([9], page 2)**.** A point $t \in \mathbb{T}$ is said to be *left-dense* if $t > \inf \mathbb{T}$ and $\varrho(t) = t$; *right-dense* if $t < \sup \mathbb{T}$ and $\sigma(t) = t$, and *dense* if t is both right-dense and left-dense; *left-scattered* if $\varrho(t) < t$; *right-scattered* if $\sigma(t) > t$, and *isolated* if t is both right-scattered and left-scattered.

Points on the time scale $\mathbb{T}$ are illustrated by Figure 1.1.

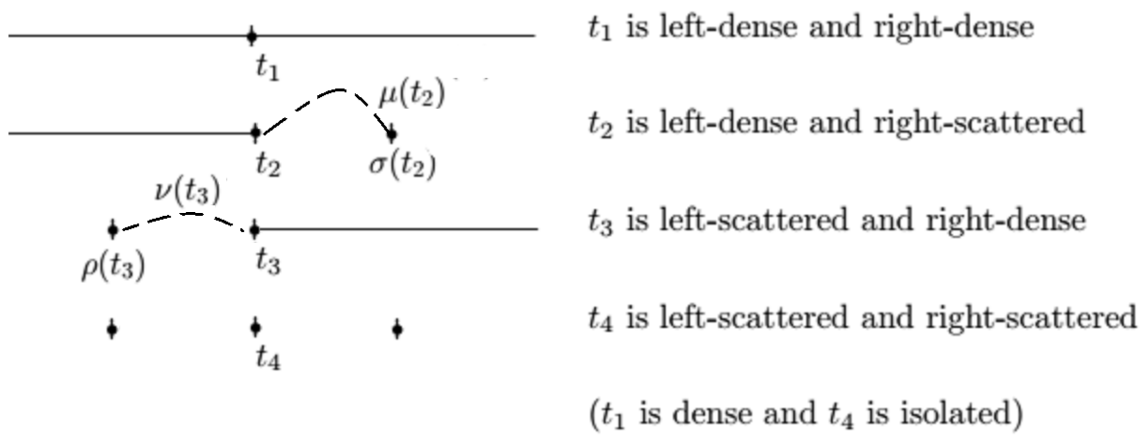

Figure 1.1: Points on the time scale $\mathbb{T}$, [9, Figure 1.1, page 2]

Note that in Definition 1.2, both $\sigma(t)$ and $\varrho(t)$ are in $\mathbb{T}$ for all $t \in \mathbb{T}$, since $\mathbb{T}$ is a closed subset of $\mathbb{R}$. We also need to define the set $\mathbb{T}^\kappa$ as follows: If $\mathbb{T}$ has a left-scattered maximum M, then $\mathbb{T}^\kappa = \mathbb{T} \backslash \{M\}$. Otherwise, $\mathbb{T}^\kappa = \mathbb{T}$. In summary,

$$\mathbb{T}^\kappa = \begin{cases} \mathbb{T} \backslash (\varrho(\sup \mathbb{T}), \sup \mathbb{T}] & \text{if } \sup \mathbb{T} < \infty, \\ \mathbb{T} & \text{if } \sup \mathbb{T} = \infty. \end{cases}$$

Next, if $f : \mathbb{T} \to \mathbb{R}$ is a function, then we define the function $f_\sigma : \mathbb{T} \to \mathbb{R}$ by $f_\sigma(t) = f(\sigma(t))$ for all $t \in \mathbb{T}$, i.e., $f_\sigma = f \circ \sigma$. Finally, we fix $t_0 \in \mathbb{T}$ and define $\mathbb{T}_{t_0} := [t_0, \infty) \cap \mathbb{T}$.

Remark 1.5. i) If $\mathbb{T} = \mathbb{R}$, then we have for any $t \in \mathbb{R}$,

$$\sigma(t) = \inf\{s \in \mathbb{R} : s > t\} = \inf(t, \infty) = t$$

and, similarly $\varrho(t) = t$. The graininess function $\mu(\cdot)$ turns out to be $\mu(t) \equiv 0$, for all $t \in \mathbb{R}$. It is easy to see that $\sigma(t) = t = \varrho(t)$. Hence, every point $t \in \mathbb{R}$ is a dense point, and $\mathbb{R}$ is a continuous time scale.

ii) Let h be a constant, and choose $\mathbb{T} = h\mathbb{Z}$, i.e.,

$$h\mathbb{Z} = \{hz : z \in \mathbb{Z}\} = \{\ldots -3h, -2h, -h, 0, h, 2h, 3h \ldots\}.$$

For all $t \in h\mathbb{Z}$ we have $\sigma(t) = t + h$, $\varrho(t) = t - h$ and $\mu(t) \equiv h$. Since $\varrho(t) < t < \sigma(t)$, t is isolated, and $h\mathbb{Z}$ is a discrete time scale.

Example 1.6. Let $a, b > 0$ be the fixed real numbers. We define the time scale denoted by $\mathbb{P}_{a,b}$ and

$$\mathbb{P}_{a,b} := \cup_{k=0}^{\infty} [k(a+b), k(a+b) + a].$$

For all $t \in \mathbb{P}_{a,b}$ we have

$$\sigma(t) = \begin{cases} t & \text{if } t \in \cup_{k=0}^{\infty} [k(a+b), k(a+b) + a), \\ t + b & \text{if } t \in \cup_{k=0}^{\infty} \{k(a+b) + a\}. \end{cases}$$

$$\varrho(t) = \begin{cases} t & \text{if } t \in \cup_{k=0}^{\infty} (k(a+b), k(a+b) + a), \\ t - b & \text{if } t \in \cup_{k=0}^{\infty} \{k(a+b)\}. \end{cases}$$

$$\mu(t) = \begin{cases} 0 & \text{if } t \in \cup_{k=0}^{\infty} [k(a+b), k(a+b) + a), \\ b & \text{if } t \in \cup_{k=0}^{\infty} \{k(a+b) + a\}. \end{cases}$$

Throughout this dissertation we make blankets assumption that t_1 and t_2 are in $\mathbb{T}$. Usually, we assume that $t_1 \le t_2$. Then the closed interval $[t_1, t_2]_{\mathbb{T}}$ in $\mathbb{T}$ defined by $[t_1, t_2]_{\mathbb{T}} := \{t \in \mathbb{T} : t_1 \le t \le t_2\}$, the open and half-open intervals etc. are defined accordingly. Note that, $[t_1, t_2]^{\kappa} = [t_1, t_2]_{\mathbb{T}}$ if t_2 is left-dense and, $[t_1, t_2)^{\kappa} = [t_1, t_2)_{\mathbb{T}} = [t_1, \varrho(t_2)]$ if t_2 is left-scattered.

1.1.2 Differentiation

We consider a function $f : \mathbb{T} \to \mathbb{R}$ and define the derivative of f on $\mathbb{T}$.

Definition 1.7 (Delta derivative, [9], page 5)**.** A function f is called *delta differentiable* at $t \in \mathbb{T}$ if there exists a function $f^{\Delta}(t)$ such that for all $\varepsilon > 0$,

$$|f(\sigma(t)) - f(s) - f^{\Delta}(t)(\sigma(t) - s)| \leq \varepsilon|\sigma(t) - s|,$$

for all $s \in U = (t - \delta, t + \delta) \cap \mathbb{T}$ and for some $\delta > 0$. The function $f^{\Delta}(t)$ is called the *delta* (or *Hilger*) *derivative* of f at the point t.

We also say that f is *delta* (or *Hilger*) *differentiable* on $\mathbb{T}^{\kappa}$ provided $f^{\Delta}(t)$ exists for all $t \in \mathbb{T}^{\kappa}$, and use words *derivative, differentiable* to replace words *delta derivative, delta differentiable* if it is not confused.

Example 1.8. Let $\mathbb{T}$ be an arbitrary time scale.

i) If $f : \mathbb{T} \to \mathbb{R}$ is defined by $f(t) = c, g(t) = t$ for all $t \in \mathbb{T}$ and $c \in \mathbb{R}$ is a constant, then it is easy to prove that $f^{\Delta}(t) = 0, g^{\Delta}(t) = 1$.

ii) If $f : \mathbb{T} \to \mathbb{R}$ is defined by $f(t) = t^2$ for all $t \in \mathbb{T}$, then $f^{\Delta}(t) = t + \sigma(t)$. Indeed, for any $\varepsilon > 0$, for all $s \in U$, and choose $\delta = \varepsilon$. Hence, we get

$$|[f(\sigma(t)) - f(s)] - [t + \sigma(t)][\sigma(t) - s]| = |t - s||\sigma(t) - s| \leq \varepsilon|\sigma(t) - s|.$$

iii) If $f : \mathbb{T} \to \mathbb{R}$ is defined by $f(t) = \frac{1}{t}$ for all $t \in \mathbb{T}, t \neq 0$, then it is not difficult to verify that $f^{\Delta}(t) = -\frac{1}{t\sigma(t)}$.

Remark 1.9. We have two following special time scales.

i) If $\mathbb{T} = \mathbb{R}$, then $f : \mathbb{R} \to \mathbb{R}$ is delta differentiable at $t \in \mathbb{T}^{\kappa} = \mathbb{T}$ if and only if there exists the limit $\lim_{s \to t} \frac{f(t)-f(s)}{t-s}$, i.e., f is differentiable (in the ordinary sense) at t, and $f^{\Delta}(t) = \lim_{s \to t} \frac{f(t)-f(s)}{t-s} = f'(t)$.

ii) If $\mathbb{T} = \mathbb{Z}$, then $f : \mathbb{Z} \to \mathbb{R}$ is delta differentiable at $t \in \mathbb{Z}$ and

$$f^{\Delta}(t) = \frac{f(\sigma(t)) - f(s)}{\mu(t)} = \frac{f(t+1) - f(t)}{1} = \Delta f(t).$$

In this case Δ is the well-known *forward difference operator* in the context of difference equations.

Next, the following theorem gives us rules for derivatives of sum, product, and quotient of delta differentiable functions.

Theorem 1.10 ([10], page 3). *Assume $f, g : \mathbb{T} \to \mathbb{R}$ are differentiable at $t \in \mathbb{T}^\kappa$:*

i) *The sum $f + g : \mathbb{T} \to \mathbb{R}$ is differentiable at t, and $(f+g)^\Delta(t) = f^\Delta(t) + g^\Delta(t)$.*

ii) *For any constant c, $cf : \mathbb{T} \to \mathbb{R}$ is differentiable at t, and $(cf)^\Delta(t) = cf^\Delta(t)$.*

iii) *The product $fg : \mathbb{T} \to \mathbb{R}$ is differentiable at t, and*

$$(fg)^\Delta(t) = f^\Delta(t)g(t) + f_\sigma(t)g^\Delta(t) = f(t)g^\Delta(t) + f^\Delta(t)g_\sigma(t).$$

iv) *If $g(t)g(\sigma(t)) \neq 0$ then $\frac{f}{g}$ is differentiable at t, and*

$$\left(\frac{f}{g}\right)^\Delta(t) = \frac{f^\Delta(t)g(t) - f_\sigma(t)g^\Delta(t)}{g(t)g_\sigma(t)}.$$

Example 1.11. Let $\mathbb{T} = h\mathbb{Z}$, and $f, g : \mathbb{T} \to \mathbb{Z}$ are the functions defined by $f(t) = t^2$, $g(t) = t^2 + 1$. We have $f^\Delta(t) = g^\Delta(t) = t + \sigma(t) = 2t + h$. Therefore, by Theorem 1.10.iii)

$$\begin{aligned}(f(t)g(t))^\Delta &= (2t+h)(t^2+1) + (t+h)^2(2t+h) \\ &= 4t^3 + 6ht^2 + 2(2h+1)t + h(h^2+1).\end{aligned}$$

Now we asssume that $(f \circ g)^\Delta(t) = f^\Delta(g(t))g^\Delta(t)$. We have

$$(f \circ g)^\Delta(t) = [2(t^2+1) + h](2t+h) = 4t^3 + 2ht^2 + 2(h+2)t + h(h+2).$$

On the other hand, $(f \circ g)(t) = f(g(t)) = (t^2+1)^2 = g^2(t)$, hence

$$(f \circ g)^\Delta(t) = [(t^2+1)^2]^\Delta = 4t^3 + 6ht^2 + 4(h^2+1)t + h(h^2+2).$$

Therefore, when $h = 1$, we have

$$(f(t)g(t))^\Delta = 4t^3 + 6t^2 + 6t + 3,$$

and $(f \circ g)^\Delta(t) = 4t^3 + 2t^2 + 6t + 3$, or $(f \circ g)^\Delta(t) = 4t^3 + 6t^2 + 8t + 3$. We then obtain $4t^2 + 2t = 0 \Rightarrow t = 0, t = -\frac{1}{2}$. It implies that the equality $(f \circ g)^\Delta(t) = f^\Delta(g(t))g^\Delta(t)$ is only true at point $t = 0, t = \frac{1}{2}$.

From the above example, we can conclude, in general, that

$$(f \circ g)^{\Delta}(t) \neq f^{\Delta}(g(t))g^{\Delta}(t).$$

The following rule will help us calculate the delta derivative of the function $(f \circ g)(t)$ exactly in the close interval $[t, \sigma(t)]$ and it is called the *chain rule*.

Theorem 1.12 (1st chain rule, [9], page 31)**.** *Assume $g : \mathbb{R} \to \mathbb{R}$ is continuous, $g : \mathbb{T} \to \mathbb{R}$ is delta differentiable on $\mathbb{T}^{\kappa}$, and $f : \mathbb{R} \to \mathbb{R}$ is continuously differentiable. Then, there exists a constant c in the real interval $[t, \sigma(t)]$ with*

$$(f \circ g)^{\Delta}(t) = f'(g(c))g^{\Delta}(t).$$

Example 1.13. Let $\mathbb{T} = h\mathbb{Z}$, $f(x) = (x+1)^2$, and $g(t) = 2t$. Find the constant c by Theorem 1.12. We have

$$\begin{aligned}
&f'(x) = 2(x+1), g^{\Delta}(t) = 2 \to f'(g(c))g^{\Delta}(t) = 4(2c+1);\\
&(f \circ g)(t) = (2t+1)^2 = 4t^2 + 4t + 1 \to (f \circ g)^{\Delta}(t) = 4(2t+h) + 4.
\end{aligned}$$

For every $t = hz \in [t_0, t_0 + h], z \in \mathbb{Z}$, we obtain $c = \frac{1}{2}h(2z+1)$. Therefore $(f \circ g)^{\Delta}(t) = f'(g(c))g^{\Delta}(t)$ for all $t \in [t_0, t_0 + h]$.

The following rule is called the second chain rule to calculate the derivative of the function $f \circ g$ on the time scale $\mathbb{T}$.

Theorem 1.14 (2nd chain rule, [9], page 32)**.** *Let $f : \mathbb{R} \to \mathbb{R}$ be continuously differentiable, and suppose that $g : \mathbb{T} \to \mathbb{R}$ is delta differentiable. Then $f \circ g : \mathbb{T} \to \mathbb{R}$ is delta differentiable and the following equality holds*

$$(f \circ g)^{\Delta}(t) = \left\{ \int_0^1 f'(g(t) + \mu(t)g^{\Delta}(t)\tau)d\tau \right\} g^{\Delta}(t).$$

To this end, we present a version of L'Hôpital's rule. Denote

$$\overline{\mathbb{T}} := \mathbb{T} \cup \{\sup \mathbb{T}\} \cup \{\inf \mathbb{T}\}.$$

If $\infty \in \overline{\mathbb{T}}$, we call ∞ left-dense, and $-\infty$ is called right-dense provided $-\infty \in \overline{\mathbb{T}}$. For any left-dense $t_0 \in \mathbb{T}$ and $\varepsilon > 0$, the set

$$L_{\varepsilon}(t_0) := \{t \in \mathbb{T} : 0 < t - t_0 \leq \varepsilon\}$$

is nonempty, and so is $L_{\infty} := \{t \in \mathbb{T} : t > \frac{1}{\varepsilon}\}$ if $\infty \in \overline{\mathbb{T}}$. The sets $R_{\varepsilon}(t_0)$ for right-dense $t_0 \in \mathbb{T}$ and $\varepsilon > 0$ are defined accordingly.

Theorem 1.15 (L'Hôpital's Rule, [9], page 49)**.** *Assume f and g are differentiable on $\mathbb{T}$ with $\lim_{t\to t_0^-} g(t) = \infty$ for some left-dense $t_0 \in \overline{\mathbb{T}}$. Suppose there exists $\varepsilon > 0$ with $g(t) > 0$, $g^\Delta(t) > 0$ for all $t \in L_\varepsilon(t_0)$. Then,*

$$\lim_{t\to t_0^-} \frac{f^\Delta(t)}{g^\Delta(t)} = r \in \overline{\mathbb{R}} \quad \textit{implies} \quad \lim_{t\to t_0^-} \frac{f(t)}{g(t)} = r.$$

1.1.3 Integration

In this section, we only present the most basic definitions and characteristics of the integral on time scale. We can learn more about this content in [9, 32]. Firstly, to describe classes of functions that are "integrable" on time scale $\mathbb{T}$, we introduce the following concepts.

Definition 1.16 ([9], page 22)**.** A function $f : \mathbb{T} \to \mathbb{R}$ is called

i) *regulated* provided its right-side limits exist (finite) at all right-dense points and its left-side limits exist (finite) at all left-dense points, in $\mathbb{T}$;

ii) *rd-continuous* provided it is continuous at right-dense points and its left-sided limits exist (finite) at left-dense points, in $\mathbb{T}$.

The set of *rd*-continuous functions $f : \mathbb{T} \to \mathbb{R}$ will be denoted by

$$\mathrm{C_{rd}} = \mathrm{C_{rd}}(\mathbb{T}) = \mathrm{C_{rd}}(\mathbb{T}, \mathbb{R}).$$

The set of functions $f : \mathbb{T} \to \mathbb{R}$ that are differentiable and whose derivative is *rd*-continuous will be denoted by

$$\mathrm{C^1_{rd}} = \mathrm{C^1_{rd}}(\mathbb{T}) = \mathrm{C^1_{rd}}(\mathbb{T}, \mathbb{R}).$$

The set of *rd*-continuous functions defined on the interval J and valued in X is denoted by $\mathrm{C_{rd}}(\mathrm{J}, \mathrm{X})$. Some results concerning *rd*-continuous and regulated functions are contained in the following theorem.

Definition 1.17 ([9], page 22)**.** A continuous function $f : \mathbb{T} \to \mathbb{R}$ is called *pre-differentiable* with (region of differentiation) D, provided $D \subset \mathbb{T}^\kappa$, $\mathbb{T}^\kappa \backslash D$ is countable and contains no right-scattered element of $\mathbb{T}$, and f is differentiable at each $t \in D$.

Theorem 1.18 ([9], page 23). *Let $f, g : \mathbb{T} \to \mathbb{R}$ be both pre-differentiable in $D \subset \mathbb{T}^\kappa$. Then $|f^\Delta(t)| \leq g^\Delta(t)$, for all $t \in D$, implies $|f(s) - f(r)| \leq g(s) - g(r)$, for all $r, s \in \mathbb{T}$, $r \leq s$.*

Theorem 1.19 ([9], page 26). *Let f be regulated. Then there exists a function F which is pre-differentiable with region of differentiation D such that $F^\Delta(t) = f(t)$ holds for all $t \in D$.*

Definition 1.20 ([32]). Assume $f : \mathbb{T} \to \mathbb{R}$ is a regulated function.

i) Any function F as in Theorem 1.19 is called a *pre-antiderivative* of f.

ii) The indefinite integral of a regulated function f is defined by

$$\int f(t)\Delta t = F(t) + C,$$

where C is an arbitrary constant and F is a pre-antiderivative of f.

iii) The *Cauchy integral* of a regulated function f is defined as following

$$\int_a^b f(t)\Delta t = F(b) - F(a) \;\text{ for all }\; a, b \in \mathbb{T},$$

where F is a pre-antiderivative of the function f.

iv) A function $F : \mathbb{T} \to \mathbb{R}$ is called an *antiderivative* of $f : \mathbb{T} \to \mathbb{R}$ provided that $F^\Delta(t) = f(t)$ holds for all $t \in \mathbb{T}^\kappa$.

Remark 1.21. i) If $\mathbb{T} = \mathbb{R}$, then

$$\int_a^b f(t)\Delta t = \int_a^b f(t)dt,$$

where f is an integrable function on a closed interval $[a, b] \subset \mathbb{R}$.

ii) If $\mathbb{T} = h\mathbb{Z}$, then

$$\int_a^b f(t)\Delta t = \begin{cases} \sum_{k=\frac{a}{h}}^{\frac{b}{h}-1} f(hk)h & \text{if } a < b, \\ 0 & \text{if } a = b, \\ -\sum_{k=\frac{b}{h}}^{\frac{a}{h}-1} f(hk)h & \text{if } a > b, \end{cases}$$

where $f : h\mathbb{Z} \to \mathbb{R}$ is an arbitrary function.

If $h = 1$, then $\mathbb{T} = \mathbb{Z}$ and

$$\int_a^b f(t)\Delta t = \begin{cases} \sum_{t=a}^{b-1} f(t) & \text{if } a < b, \\ 0 & \text{if } a = b, \\ -\sum_{t=b}^{a-1} f(t) & \text{if } a > b, \end{cases}$$

Example 1.22. i) Let $\mathbb{T} = h\mathbb{Z}$ and a is a constant, $a \neq 1$. For all $t \in \mathbb{T}$, since

$$\left(\frac{a^t}{a^h - 1}\right)^{\Delta} = \Delta\left(\frac{a^t}{a^h - 1}\right) = \frac{a^{t+h} - a^t}{a^h - 1} = a^t,$$

we have

$$\int a^t \Delta t = \frac{a^t}{a^h - 1} + C,$$

where C is an arbitrary constant.

ii) Let $\mathbb{T} = \mathbb{Z}$ and $\alpha \in \mathbb{R}$. Since

$$\binom{t}{\alpha+1}^{\Delta} = \binom{t+1}{\alpha+1} - \binom{t}{\alpha+1} = \binom{t}{\alpha},$$

we get

$$\int \binom{t}{\alpha}\Delta t = \binom{t}{\alpha+1} + C.$$

Theorem 1.23 ([10], page 8)**.** *Every rd-continuous function has an antiderivative. In particular if $t_0 \in \mathbb{T}$, then F defined by*

$$F(t) := \int_{t_0}^{t} f(\tau)\Delta\tau,$$

for all $t \in \mathbb{T}$, is an antiderivative of f.

We now present some useful properties of integral on time scales.

Theorem 1.24 ([10], page 8)**.** *If $a, b \in \mathbb{T}, \alpha \in \mathbb{R}$ and $f, g \in C_{rd}(\mathbb{T}, \mathbb{R})$, then*

i) $\displaystyle\int_t^{\sigma(t)} f(\tau)\,\Delta\tau = f(t)\mu(t)$, *with $t \in \mathbb{T}^{\kappa}$;*

ii) $\displaystyle\int_a^b f(\sigma(t))g^{\Delta}(t)\,\Delta t = (fg)(b) - (fg)(a) - \int_a^b f^{\Delta}(t)g(t)\,\Delta t;$

iii) $\displaystyle\int_a^b f(t)g^{\Delta}(t)\,\Delta t = (fg)(b) - (fg)(a) - \int_a^b f^{\Delta}(t)g_{\sigma}(t)\,\Delta t;$

iv) *If* $f(t) \geq 0$, *for all* $t \in [a,b)$, *then* $\int_a^b f(t)\,\Delta t \geq 0$;

v) *If* $|f(t)| \leq g(t)$, *for all* $t \in [a,b)$, *then* $\left|\int_a^b f(t)\,\Delta t\right| \leq \int_a^b g(t)\Delta t$;

vi) *If* $f^{\Delta} \geq 0$, *then* f *is increasing.*

We will use the following result later. By $f^{\Delta}(t,\tau)$ in the following theorem, we mean for each fixed τ the derivative of $f(t,\tau)$ with respect to t.

Theorem 1.25 ([9], page 46)**.** *Let* $a \in \mathbb{T}^{\kappa}, b \in \mathbb{T}$ *and suppose* $f : \mathbb{T} \times \mathbb{T}^{\kappa} \to \mathbb{R}$ *is continuous at* (t,t)*, where* $t \in \mathbb{T}^{\kappa}, t > a$*. Also assume that* $f^{\Delta}(t,\cdot)$ *is rd-continuous on the interval* $[a,\sigma(t)]$*. Suppose that for every* $\varepsilon > 0$*, there exists a neighbourhood* U *of* t *such that*

$$|f(\sigma(t),\tau) - f(s,\tau) - f^{\Delta}(t,\tau)(\sigma(t)-s)| \leq \varepsilon|\sigma(t)-s|,$$

for all $s \in U$*, where* f^{Δ} *denotes the derivative of* f *with respect to the first variable. Then*

$$g(t) := \int_a^t f(t,\tau)\Delta\tau \ \textit{ implies } \ g^{\Delta}(t) = f(\sigma(t),t) + \int_a^t f^{\Delta}(t,\tau)\Delta\tau.$$

1.1.4 Regressivity

Definition 1.26 ([10], page 10)**.** A function $p : \mathbb{T} \to \mathbb{R}$ is called

i) *regressive*, if $1 + \mu(t)p(t) \neq 0$, for all $t \in \mathbb{T}^{\kappa}$;

ii) *positively regressive*, if $1 + \mu(t)p(t) > 0$, for all $t \in \mathbb{T}^{\kappa}$;

iii) *uniformly regressive*, if there exists a number $\delta > 0$ such that

$$|1 + \mu(t)p(t)| \geq \delta, \text{ for all } t \in \mathbb{T}^{\kappa}.$$

In case of constant function p, we also say that p is a regressive number. Denote $\mathcal{R} = \mathcal{R}(\mathbb{T},\mathbb{R})$ (resp., $\mathcal{R}^+ = \mathcal{R}^+(\mathbb{T},\mathbb{R})$) the set of the regressive (resp., positively regressive) functions on time scale $\mathbb{T}$.

By Definition 1.3, if $p,q \in \mathcal{R}$, it is easy to verify that $p \oplus q, p \ominus q, \ominus p, \ominus q \in \mathcal{R}$. The element $(\ominus q)(\cdot)$ is called the inverse element of $q(\cdot)$. It can be directly seen that, the set $\mathcal{R}(\mathbb{T}, \mathbb{R})$ with the operator $\oplus$ forms an Abelian group, and called the *regressive group*. In addition, the set $\mathcal{R}^+(\mathbb{T}, \mathbb{R})$ is a subgroup of $\mathcal{R}(\mathbb{T}, \mathbb{R})$. We now define the regressivity of $n \times n$-matrix functions and state some related results.

Definition 1.27 ([9], page 190)**.** An $n \times n$-matrix $A(\cdot)$ defined on time scale $\mathbb{T}$ is called regressive if the matrix $I + \mu(t)A(t)$ is invertible for all $t \in \mathbb{T}^\kappa$, i.e., the $\det(I + \mu(t)A(t)) \neq 0$, where $I = I_n$ is the identity matrix in $\mathbb{R}^{n\times n}$. The set of all regressive matrices is denoted by $\mathcal{R}(\mathbb{T}, \mathbb{R}^{n\times n})$.

Lemma 1.28 ([9], page 190)**.** *Matrix $A(t), t \in \mathbb{T}$ is regressive if and only if the eigenvalues $\lambda_i(t)$ of $A(t)$ are regressive for all $1 \leq i \leq n$.*

Definition 1.29 ([9], page 191)**.** For all $t \in \mathbb{T}^\kappa$ and $A(t), B(t)$ in $\mathcal{R}(\mathbb{T}, \mathbb{R}^{n\times n})$ we define the following operators

i) $(A \oplus B)(t) = A(t) + B(t) - \mu(t)A(t)B(t)$;

ii) $(\ominus A)(t) = -[I + \mu(t)A(t)]^{-1}A(t) = -A(t)[I + \mu(t)A(t)]^{-1}$;

iii) $(A \ominus B)(t) = A(t) - [I + \mu(t)B(t)]^{-1}B(t)$.

It is clear that the set $\mathcal{R}(\mathbb{T}, \mathbb{R}^{n\times n})$ is a group with the circle plus $\oplus$. This group is not Abelian, because $(A \oplus B)(t) \neq (B \oplus A)(t)$. Note that, if $AB = BA$ then $(A \oplus B)(t) = (B \oplus A)(t)$.

1.2 Exponential Function

We firstly introduce the *Hilger complex plane*.

Definition 1.30 ([9], page 51)**.** For $h \in \mathbb{R}, h > 0$, the *Hilger complex numbers*, the *Hilger axis*, the *Hilger imaginary circle* and the *Hilger alternating axis* are defined as follows,

$$\mathbb{C}_h := \left\{ z \in \mathbb{C} : z \neq -\frac{1}{h} \right\}, \qquad \mathbb{R}_h := \left\{ z \in \mathbb{C}_h : z \in \mathbb{R}, \text{ and } z > -\frac{1}{h} \right\},$$
$$\mathbb{I}_h := \left\{ z \in \mathbb{C} : \left| z + \frac{1}{h} \right| = \frac{1}{h} \right\}, \quad \mathbb{A}_h := \left\{ z \in \mathbb{C}_h : z \in \mathbb{R}, \text{ and } z < -\frac{1}{h} \right\},$$

respectively. If $h = 0$, we set $\mathbb{C}_0 = \mathbb{C}, \mathbb{R}_0 = \mathbb{R}, \mathbb{I}_0 = i\mathbb{R}$ and $\mathbb{A}_0 = \emptyset$.

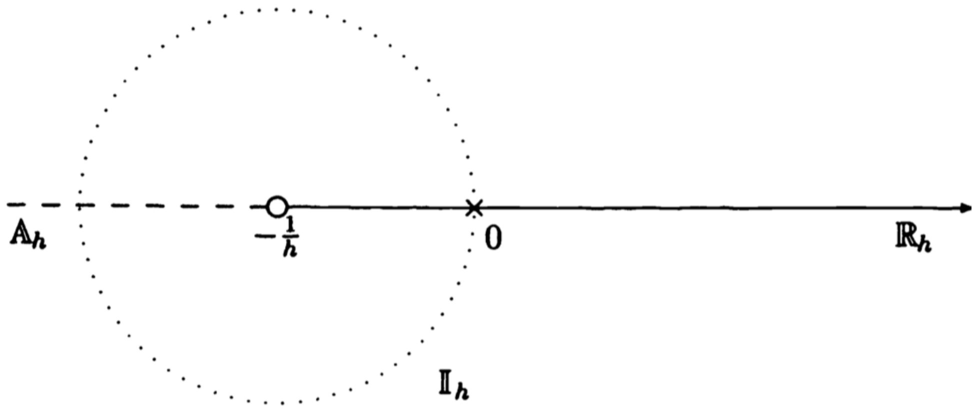

Figure 1.2: Hilger's Complex Plane, [9, Figure 2.1, page 52]

The sets introduced in Definition 1.30 for h > 0 are illustrated in Figure 1.2.

Definition 1.31 ([9], page 52)**.** Let $h > 0$. For any $z \in \mathbb{C}_h$, we define

i) the *Hilger real* part of z by $\Re_h(z) := \dfrac{|zh+1|-1}{h}$;

ii) the *Hilger imaginary* part of z by $\Im_h(z) := \dfrac{\operatorname{Arg}(zh+1)}{h}$,
where $\operatorname{Arg}(z)$ denotes the principal argument of z, $-\pi < \operatorname{Arg}(z) \leq \pi$.

For $h > 0$, let $\mathbb{Z}_h$ be the strip

$$\mathbb{Z}_h(z) := \left\{ z \in \mathbb{C} : -\frac{\pi}{h} < \Im(z) \leq \frac{\pi}{h} \right\},$$

and for $h = 0$, then $\mathbb{Z}_0 := \mathbb{C}$.

Definition 1.32 ([9], page 57)**.** For $h \geq 0$ and $z \in \mathbb{C}_h$, the *cylinder transformation* $\xi : \mathbb{C}_h \to \mathbb{Z}_h$ is defined by

$$\xi_h(z) := \begin{cases} \dfrac{\operatorname{Log}(1+zh)}{h} & \text{if } h > 0, \\ z & \text{if } h = 0, \end{cases}$$

Log is the principal logarithm function with the value region $[-i\pi, i\pi)$, i.e.,

$$\xi_h(z) := \frac{1}{h} \begin{cases} \log(1+zh) & \text{for } z > -\frac{1}{h}, \\ \log|1+zh| + i\pi & \text{for } z < -\frac{1}{h}. \end{cases}$$

We now use the cylinder transformation to define a generalized exponential function $e_p(t,s)$ for an arbitrary time scale $\mathbb{T}$.

Definition 1.33 ([9], page 59)**.** If $p(\cdot) \in \mathcal{R}$, then the exponential function on the time scale $\mathbb{T}$ is define by

$$e_p(t,s) = \exp\left(\int_s^t \xi_{\mu(\tau)}(p(\tau))\Delta\tau\right) \text{ for all } s,t \in \mathbb{T}. \tag{1.1}$$

It is easy to verify that if $p(\cdot) \in \mathcal{R}(\mathbb{T},\mathbb{R})$, then the semigroup property is satisfied, i.e., $e_p(t,r)e_p(r,s) = e_p(t,s)$ for all $r,s,t \in \mathbb{T}$.

Theorem 1.34 ([9], page 62)**.** *Given $p(\cdot), q(\cdot) \in \mathcal{R}$, for all $s,t \in \mathbb{T}$, we have*

i) $e_p(t,t) = 1,\ e_0(t,s) = 1$;

ii) $e_p(\sigma(t),s) = (1+\mu(t)p(t))e_p(t,s)$;

iii) $e_p(t,s)e_q(t,s) = e_{p\oplus q}(t,s)$;

iv) $\dfrac{e_p(t,s)}{e_q(t,s)} = e_{p\ominus q}(t,s)$, *and* $e_p(t,s) = \dfrac{1}{e_p(s,t)} = e_{\ominus p}(s,t)$;

v) *If* $p(\cdot) \in \mathcal{R}^+$ *then* $e_p(t,s) > 0$;

vi) *If* $p(\cdot), q(\cdot) \in \mathcal{R}^+$, $p \le q$ *then* $0 < e_p(t,s) \le e_q(t,s)$, *for all* $t \ge s$;

vii) $[e_p(\cdot,s)]^\Delta(t) = p(t)e_p(t,s)$, $[e_p(t,\cdot)]^\Delta(s) = \ominus p(t)e_p(t,s)$;

viii) $\left(\dfrac{1}{e_p(\cdot,s)}\right)^\Delta(t) = -\dfrac{p(t)}{e_p(\sigma(t),s)}$.

Example 1.35. We present some exponential functions on several special time scales.

i) If $\mathbb{T} = \mathbb{R}$, then $e_p(t,s) = \exp\left(\int_s^t p(\tau)d\tau\right)$, and if $p(\cdot)$ is a constant function, then $e_p(t,s) = e^{p(t-s)}$.

ii) If $\mathbb{T} = h\mathbb{Z}$, then $e_p(t,s) = \Pi_{\tau=s}^t(1+hp(\tau))$, and if $p(\cdot)$ is a constant function, we have $e_p(t,s) = (1+hp)^{\frac{t-s}{h}}$.

iii) Considering the time scale $\mathbb{P}_{a,b}$ as in Example 1.6 with $a = b = 1$, i.e.,

$$\mathbb{P}_{1,1} = \cup_{k=0}^{\infty}[2k, 2k+1].$$

If α is a given constant, then we obtain the exponential function

$$e_{\alpha}(t,0) = \left(\frac{1+\alpha}{e^{\alpha}}\right)^{[\frac{t}{2}]} e^{\alpha t}, \text{ for all } t \in \mathbb{P}_{1,1}.$$

1.3 Dynamic Inequalities

We consider some inequalities on time scale. From the definition of $\xi_h(x)$ we see that $|\xi_h(x)| \leq |x|$. Therefore, by (1.1) it follows that

$$0 < e_{\alpha}(t, t_0) \leq e^{\alpha(t-t_0)},$$

for any positive regressive number α. Next, we consider some important inequalities on time scales.

1.3.1 Gronwall's Inequality

This is an important inequality and we only consider the most basic forms used as an necessary object to prove many results obtained in this thesis.

Lemma 1.36 (Gronwall-Bellman's Lemma, [9], page 257). *Let $y \in C_{rd}(\mathbb{T}, \mathbb{R})$ and $k \in \mathcal{R}^{+}(\mathbb{T}, \mathbb{R})$, $k \geq 0$, $\alpha \in \mathbb{R}$. Assume that $y(t)$ satisfies the inequality*

$$y(t) \leq \alpha + \int_{t_0}^{t} k(s)y(s)\Delta s, \text{ for all } t \in \mathbb{T}, t \geq t_0.$$

Then, the relation $y(t) \leq \alpha e_{k(t)}(t, t_0)$ holds for all $t \in \mathbb{T}, t \geq t_0$.

Corollary 1.37 ([60]). *Let nonegative functions $y, f, k \in C_{rd}(\mathbb{T}, \mathbb{R}^{+})$, and the function f be nondecreasing on $\mathbb{T}$. If the estimate*

$$y(t) \leq f(t) + \int_{t_0}^{t} k(s)y(s)\,\Delta s \text{ for all } t \in \mathbb{T}$$

is satisfied, then $y(t) \leq f(t)e_{k(t)}(t, t_0)$ holds for all $t \in \mathbb{T}$.

Theorem 1.38 ([1], Theorem 3.5). *Let $\tau \in \mathbb{T}$, $a \in \mathcal{R}^+$ and $u, b \in C_{rd}(\mathbb{T}, \mathbb{R})$. Then $u^\Delta(t) \geq -a(t)u_\sigma(t) + b(t)$, for all $t \geq \tau$ implies*

$$u(t) \geq u(\tau)e_{\ominus a}(t,\tau) + \int_\tau^t b(s)e_{\ominus a}(t,s)\Delta s, \text{ for all } t \geq \tau.$$

We can study more about Gronwall-type inequalities in [1, 60].

1.3.2 Hölder's and Minkowskii's Inequalities

Theorem 1.39 (Hölder's Inequality, [9], page 259). *Let $a, b \in \mathbb{T}$. For rd-continuous functions $f, g : [a, b] \to \mathbb{R}$ we have*

$$\int_a^b |f(t)g(t)|\Delta t \leq \left\{\int_a^b |f(t)|^p \Delta t\right\}^{\frac{1}{p}} \left\{\int_a^b |g(t)|^q \Delta t\right\}^{\frac{1}{q}},$$

where $p > 1$ and $\dfrac{1}{p} + \dfrac{1}{q} = 1$.

In case $p = q = 2$, we get Cauchy-Schwarz's inequality.

Theorem 1.40 ([9], page 260). *Let $a, b \in \mathbb{T}$. For rd-continuous functions $f, g : [a, b] \to \mathbb{R}$ we have*

$$\int_a^b |f(t)g(t)|\Delta t \leq \sqrt{\left\{\int_a^b |f(t)|^2 \Delta t\right\} \left\{\int_a^b |g(t)|^2 \Delta t\right\}}.$$

Minkowskii's inequality is also deduced from Hölder's inequality .

Theorem 1.41 ([9], page 260). *Let $a, b \in \mathbb{T}$. For rd-continuous functions $f, g : [a, b] \to \mathbb{R}$ and $p > 1$, we have*

$$\left\{\int_a^b |f(t) + g(t)|\Delta t\right\}^{\frac{1}{p}} \leq \left\{\int_a^b |f(t)|^p \Delta t\right\}^{\frac{1}{p}} + \left\{\int_a^b |g(t)|^p \Delta t\right\}^{\frac{1}{p}}.$$

1.4 Linear Dynamic Equation

Let $A : \mathbb{T}^\kappa \to \mathbb{R}^{n \times n}$ be rd-continuous. Consider n-dimensional linear dynamic equations $x^\Delta = A(t)x$ for all $t \in \mathbb{T}$.

Theorem 1.42 ([36]). *Assume that $A(\cdot)$ is a rd-continuous matrix-valued function. Then, for each $t_0 \in \mathbb{T}^\kappa$, the initial value problem*

$$x^\Delta = A(t)x, \ x(t_0) = x_0, \tag{1.2}$$

has a unique solution $x(\cdot)$ defined on $\mathbb{T}_{t_0}$. Moreover, if $A(\cdot)$ is regressive then this solution defines on $t \in \mathbb{T}^\kappa$.

The solution of Equation (1.2) is called the *Cauchy operator*, or the *matrix exponential function* and be denoted by $\Phi_A(t, t_0)$ or $\Phi(t, t_0)$. Note that, $\Phi_A(t, t_0)$ always exists for all $t \geq t_0$, even if $A(\cdot)$ is not a regressive matrix. If $A(\cdot)$ is regressive, then Cauchy operator $\Phi_A(t, t_0)$ is defined for all $t, t_0 \in \mathbb{T}^\kappa$ (see [36, 65]).

If $A(\cdot)$ is commutative with its integral $\int_{t_0}^{t} A(\tau)\Delta\tau$, then we also use $e_A(t, t_0)$ to denote $\Phi_A(t, t_0)$. Especially, if $\mathbb{T} = \mathbb{R}$ and $A(\cdot)$ is a constant matrix, then $\Phi_A(t, s) = e_A(t, s) = e^{A(t-s)}$. If $\mathbb{T} = h\mathbb{Z}$, $h \in \mathbb{N}$, and $A(\cdot)$ is a constant matrix, then $\Phi_A(t, s) = e_A(t, s) = (I + hA)^{\frac{t-s}{h}}$.

Note that, the solution $x(\cdot)$ of Equation (1.2) can be represented by the Cauchy operator, i.e.,

$$x(\cdot) = \Phi_A(\cdot, t_0).$$

Lemma 1.43 ([36]). *Suppose $t, s, \tau \in \mathbb{T}$ and $A(\cdot), B(\cdot)$ are $n \times n$-matrix functions defined on the time scale $\mathbb{T}$. Then the following statements hold true:*

i) $\Phi_A(t, s) = \Phi_A(t, \tau)\Phi_A(\tau, s)$, *is called the cocycle property;*

ii) $\Phi_A(\sigma(t), s) = (I + \mu(t)A(t))\Phi_A(t, s)$;

iii) *If $\Phi_A(t, s)$ is commutative with $B(t)$ then $\Phi_A(t, s)\Phi_B(t, s) = \Phi_{A \oplus B}(t, s)$.*

Theorem 1.44 (Variation of constants formula, [9], page 195). *Let $A : \mathbb{T}^\kappa \to \mathbb{R}^{m \times m}$ and $f : \mathbb{T}^\kappa \times \mathbb{R}^m \to \mathbb{R}^m$ be rd-continuous. If $x(t)$, $t \geq t_0$, is a solution of the dynamic equation*

$$x^\Delta = A(t)x + f(t, x), \ x(t_0) = x_0.$$

Then we have

$$x(t) = \Phi_A(t, t_0)x_0 + \int_{t_0}^{t} \Phi_A(t, \sigma(s))f(s, x(s))\Delta s, \ \textit{for all} \ t \in \mathbb{T}_{t_0}.$$

1.5 Stability of Dynamic Equation

Let $\mathbb{T}$ be a time scale, $t_0 \in \mathbb{T}$. Consider dynamic equation of the form

$$x^\Delta = f(t,x), \quad x(t_0) = x_0 \in \mathbb{R}^m, \quad t \in \mathbb{T}, \tag{1.3}$$

where $f : \mathbb{T} \times \mathbb{R}^m \to \mathbb{R}^m$ is rd-continuous. If $f(t,0) = 0$, then Equation (1.3) has the trivial solution $x \equiv 0$.

Denote by $x(t;t_0,x_0)$ the solution of Cauchy problem (1.3). Suppose that for any $x_0 \in \mathbb{R}^m$, there exists a unique solution satisfying $x(t_0;t_0,x_0) = x_0$ and this solution is defined on $\mathbb{T}_{t_0}$.

There are two concepts of exponential stability for the dynamic equations on time scales. One of them is to compare the solutions of (1.3) with exponential functions on the time scale $\mathbb{T}$, Definition 1.45, meanwhile the other is based on classical exponential functions, Definition 1.46. In fact, these definitions are equivalent to each other on any time scale that have the bounded graininess function.

Definition 1.45 ([15, 36]). The trivial solution $x \equiv 0$ of dynamic equation (1.3) is said to be exponentially stable if there exist a positive constant α with $-\alpha \in \mathcal{R}^+$ and a positive number $\delta > 0$ such that for each $t_0 \in \mathbb{T}$ there exists a number $N = N(t_0) \geq 1$ for which, the solution of (1.3) with the initial condition $x(t_0) = x_0$ satisfies

$$\|x(t;t_0,x_0)\| \leq N\|x_0\|e_{-\alpha}(t,t_0),$$

for all $t \geq t_0, t \in \mathbb{T}$ and $\|x_0\| < \delta$.

Definition 1.46 ([30, 66]). The trivial solution $x \equiv 0$ of dynamic equation (1.3) is said to be exponentially stable if there exist a positive constant α and a positive number $\delta > 0$ such that for each $t_0 \in \mathbb{T}$, there exists a number $N = N(t_0) \geq 1$ for which, the solution of (1.3) with the initial condition $x(t_0) = x_0$ satisfies

$$\|x(t;t_0,x_0)\| \leq N\|x_0\|e^{-\alpha(t-t_0)},$$

for all $t \geq t_0, t \in \mathbb{T}$ and $\|x_0\| < \delta$.

Note that, in both Definitions 1.45 and 1.46, if the constant N can be chosen independently of $t_0 \in \mathbb{T}$ then the solution $x \equiv 0$ of Equation (1.3) is called uniformly exponentially stable. Furthermore, when applying Definition 1.45, condition $-\alpha \in \mathcal{R}^+$ is equivalent to $\mu(t) < \frac{1}{\alpha}$, i.e, we are working on time scales with bounded graininess function.

Theorem 1.47 ([49]). *On the time scales with bounded graininess, Definition* 1.45 *is equivalent to Definition* 1.46.

Theorem 1.48 ([15]). *Suppose there exists a constant α, such that for all $t \in \mathbb{T}$, $\|A(t)\| \leq \alpha$. Then the time-varying linear dynamic equation* (1.2) *is uniformly exponentially stable if and only if there exists a constant $\beta > 0$ such that*

$$\int_{t_0}^{t} \|\Phi_A(t, \sigma(s))\| \Delta s \leq \beta$$

for all $t, t_0 \in \mathbb{T}$ with $t \geq \sigma(t_0)$.

Definition 1.49 ([66]). Let $\mathbb{T}$ be a time scale which is unbounded from above. System (1.2) is called *robustly exponentially stable* if there is a number $\varepsilon > 0$ such that the exponential stability of (1.2) implies the exponential stability of equation $x^\Delta = B(t)x$ for any rd-continuous function $B : \mathbb{T} \to \mathbb{K}^{n\times n}$ with $\sup_{t\in\mathbb{T}} \|B(t) - A(t)\| \leq \varepsilon$. In particular, if A is constant we call (1.2) robustly exponentially stable if all of matrices B in a suitable neighborhood of A, the corresponding system is exponentially stable.

Conclusions of Chapter 1. In this chapter, we introduce the most basic information needed throughout this thesis for the time scale analysis. The cylinder transformation is given in order to define the improper exponential function related to an rd-continuous regressive function. It is proven that this function is, in fact, a unique solution to the initial value problem of a scalar first-order linear dynamic equation. We also introduce the concepts of exponential stability and uniformly exponential stability for the dynamic equations on time scales by using classical exponential functions or improper exponential functions. Further, we show that two of these definitions are equivalent if the time scale has bounded graininess. At the end of this chapter, we also introduce the definitions of exponential stability, uniformly exponential stability and robustly exponential stability for the n-dimensional linear system of the dynamic equations with $A(\cdot)$ is an rd-continuous function $n \times n$-matrix on time scale $\mathbb{T}$.

CHAPTER 2

LYAPUNOV EXPONENTS FOR DYNAMIC EQUATIONS

In this chapter, we will study the first Lyapunov method for dynamic equations on time scales. The content of Chapter 2 is based on paper No.1 in the list of the author's scientific works.

It is well-known that it is not able to define the logarithm function on time scales (see [8]). However, the idea of comparing the growth rate of a normal function with exponential functions in the definition of the classical Lyapunov exponent is still useful on the time scales. Therefore, instead of considering the limit

$$\limsup_{t\to\infty} \frac{1}{t} \ln |f(t)|,$$

we can study the oscillation of the ratio

$$\frac{|f(t)|}{e_\alpha(t,t_0)} \text{ as } t \to \infty$$

in the parameter α to define the Lyapunov exponent of the function f on a time scale, where $e_\alpha(t,t_0)$ is the exponential function on the time scale defined in Chapter 1.

Assume that $\mathbb{T}$ is unbounded from above, i.e., $\sup \mathbb{T} = \infty$, and the graininess function $\mu(t)$ is bounded on $\mathbb{T}$, i.e., there exists a $\mu^* = \sup_{t\in\mathbb{T}} \mu(t) < \infty$. This is equivalent to the existence of positive numbers m_1, m_2 such that for every element $t \in \mathbb{T}$, there exists a quantity that depends on t, $c = c(t) \in \mathbb{T}$, satisfying the condition $m_1 \le c - t < m_2$, see [65]. Furthermore, by definition, if $\alpha \in \mathbb{R} \cap \mathcal{R}^+$ then $\alpha > -\frac{1}{\mu(t)}$ for all $t \in \mathbb{T}$. As a consequence, we have $\inf(\mathbb{R} \cap \mathcal{R}^+) = -\frac{1}{\mu^*}$, supplemented by $\frac{1}{0} = \infty$.

2.1 Lyapunov Exponent: Definition and Properties

2.1.1 Definition

Definition 2.1. The *Lyapunov exponent* of the function $f : \mathbb{T}_{t_0} \to \mathbb{K}$ is a real number $a \in \mathcal{R}^+$ such that for all arbitrary numbers $\varepsilon > 0$, we have

$$\lim_{t\to\infty} \frac{|f(t)|}{e_{a\oplus\varepsilon}(t,t_0)} = 0, \tag{2.1}$$

$$\limsup_{t\to\infty} \frac{|f(t)|}{e_{a\ominus\varepsilon}(t,t_0)} = \infty. \tag{2.2}$$

The Lyapunov exponent of the function f is denoted by $\kappa_L[f]$.

If (2.1) is true for all $a \in \mathbb{R} \cap \mathcal{R}^+$ then we say by convention that f has *left extreme exponent*, $\kappa_L[f] = -\frac{1}{\mu^*} = \inf(\mathbb{R} \cap \mathcal{R}^+)$. If (2.2) is true for all $a \in \mathbb{R} \cap \mathcal{R}^+$, we say that the function f has *right extreme exponent*, $\kappa_L[f] = +\infty$. If $\kappa_L[f]$ is neither left extreme exponent nor right extreme exponent, then we call $\kappa_L[f]$ by *normal Lyapunov exponent*.

A necessary and sufficient condition for the existence of normal Lyapunov exponent is showed in the following lemma.

Lemma 2.2. *Let $f : \mathbb{T}_{t_0} \to \mathbb{K}$ be a function. Then, f has a normal Lyapunov exponent if and only if there exist two real numbers $\lambda, \gamma \in \mathcal{R}^+$ with $\lambda \neq \inf(\mathbb{R} \cap \mathcal{R}^+)$ such that*

$$\lim_{t\to\infty} \frac{|f(t)|}{e_{\gamma}(t,t_0)} = 0, \textit{ and } \quad \limsup_{t\to\infty} \frac{|f(t)|}{e_{\lambda}(t,t_0)} = \infty. \tag{2.3}$$

Proof. Let the Lyapunov exponent $\kappa_L[f]$ be normal, this means that

$$-\frac{1}{\mu^*} < \kappa_L[f] < \infty.$$

Choose $\lambda, \gamma \in \mathcal{R}^+$, such that

$$-\frac{1}{\mu^*} < \lambda < \kappa_L[f] < \gamma < \infty.$$

For small enough $\varepsilon > 0$, we have that

$$\lambda < \kappa_L[f] \ominus \varepsilon < \kappa_L[f] \oplus \varepsilon < \gamma$$

for any $t \in \mathbb{T}_{t_0}$. From (2.1) and (2.2) we obtain (2.3).

On the contrary, we assume that there are numbers λ and γ such that the limits in (2.3) hold. We define the sets

$$\begin{aligned} A &:= \Big\{\lambda_0 \in \mathbb{R} \cap \mathcal{R}^+ : \frac{|f(t)|}{e_{\lambda_0}(t,t_0)} \text{ is unbounded on } \mathbb{T}_{t_0}\Big\}, \\ B &:= \Big\{\lambda_1 \in \mathbb{R} \cap \mathcal{R}^+ : \lim_{t\to\infty} \frac{|f(t)|}{e_{\lambda_1}(t,t_0)} = 0\Big\}. \end{aligned} \tag{2.4}$$

Since $\lambda \in A$ and $\gamma \in B$, it follows that A, B are nonempty. Furthermore, if $x \in A$ and $y \in B$ then we get $x \le y$. Hence, A is bounded from above, B is bounded from below and $\sup A < \gamma, \inf B > \lambda$. It follows directly that $\sup A = \inf B$ and this common value is denoted by the number a. For every $\varepsilon > 0$, let ε_1 be a positive number satisfying the condition $a \oplus \varepsilon \ge a + \varepsilon_1$. By the definition of number a, we have

$$\lim_{t\to\infty} \frac{|f(t)|}{e_{a\oplus\varepsilon}(t,t_0)} \le \lim_{t\to\infty} \frac{|f(t)|}{e_{a+\varepsilon_1}(t,t_0)} = 0,$$

which deduces that

$$\lim_{t\to\infty} \frac{|f(t)|}{e_{a\oplus\varepsilon}(t,t_0)} = 0.$$

In addition, by setting

$$\varepsilon_2 := \frac{\varepsilon(1 + a \inf_t \mu(t))}{1 + \varepsilon \sup_t \mu(t)} \le \frac{\varepsilon(1 + a\mu(t))}{1 + \varepsilon\mu(t)},$$

it is easy to obtain

$$a \ominus \varepsilon = \frac{a - \varepsilon}{1 + \varepsilon\mu(t)} \le a - \varepsilon_2 \in \mathcal{R}^+.$$

Hence, since $a - \varepsilon_2 \in A$,

$$\frac{|f(t)|}{e_{a\ominus\varepsilon}(t,t_0)} \ge \frac{|f(t)|}{e_{a-\varepsilon_2}(t,t_0)}$$

is unbounded from above. Thus, the number a satisfies Definition 2.1.

To prove the uniqueness, we suppose that b is a real number and satisfies the conditions (2.1) and (2.2). We will prove that $a = b$. Indeed, supposing on the contrary that $a < b$. Choose an arbitrary small number $\varepsilon > 0$, such that

$$\mu^*(1 + \mu^*|a|)\varepsilon^2 + 2(1 + \mu^*|a|)\varepsilon + (a - b) \le 0$$

or

$$a+\varepsilon+\mu^*a\varepsilon \le \frac{b-\varepsilon}{1+\mu^*\varepsilon}.$$

We have

$$a\oplus\varepsilon = a+\varepsilon+\mu(t)a\varepsilon \le a+\varepsilon+\mu^*a\varepsilon \le \frac{b-\varepsilon}{1+\mu^*\varepsilon} \le \frac{b-\varepsilon}{1+\mu(t)\varepsilon} = b\ominus\varepsilon.$$

Hence, $e_{a\oplus\varepsilon}(t,t_0) \le e_{b\ominus\varepsilon}(t,t_0)$ which implies that

$$\frac{|f(t)|}{e_{a\oplus\varepsilon}(t,t_0)} \ge \frac{|f(t)|}{e_{b\ominus\varepsilon}(t,t_0)}$$

and we have a contradiction. The proof is complete . □

Remark 2.3. i) In case $\mathbb{T}=\mathbb{R}$, Definition 2.1 leads to the classical one of the Lyapunov exponent, i.e., $\kappa_L[f]=\chi[f]=\limsup\limits_{t\to\infty}\frac{\ln|f(t)|}{t}$.

ii) In case $\mathbb{T}=\mathbb{Z}$, it is clear that $\ln(1+\kappa_L[f]) = \limsup\limits_{n\to\infty}\frac{\ln|f(n)|}{n} = \chi[f]$. Furthermore, the left extreme exponent is $\inf(\mathbb{R}\cap\mathcal{R}^+)=-1$.

2.1.2 Properties

In this subsection, we consider functions $f,g:\mathbb{T}_{t_0}\to\mathbb{K}$. We have some basic properties.

Lemma 2.4. *There hold following assertions.*

i) $\kappa_L[|f|]=\kappa_L[f]$;

ii) $\kappa_L[0]=\inf(\mathbb{R}\cap\mathcal{R}^+)$;

iii) $\kappa_L[cf]=\kappa_L[f]$, *where $c\neq 0$ is a constant;*

iv) *If $a\in\mathbb{R}\cap\mathcal{R}^+$ and* (2.1) *is satisfied for any $\varepsilon>0$ then $\kappa_L[f]\le a$. Similarly, if $a\in\mathbb{R}\cap\mathcal{R}^+$ and* (2.2) *holds for any $\varepsilon>0$ then $\kappa_L[f]\ge a$;*

v) *If $|f(t)|\le|g(t)|$ for all t large enough then $\kappa_L[f]\le\kappa_L[g]$;*

vi) *If f is bounded from above (resp. from below) then $\kappa_L[f]\le 0$ (resp. $\kappa_L[f]\ge 0$). As a consequence, if f is bounded then $\kappa_L[f]=0$.*

Proof. We can directly see that the proofs of statements i), ii) and iii) are deduced from Definition 2.1. We now prove the properties iv), v) and vi).

iv) By the assumption, $a \in A$ where A is defined by (2.4). Thus, $a \geq \inf A = \kappa_L[f]$. If λ is the left extreme exponent then we have $\lambda = \kappa_L[f] \leq a$. If λ is not the left extreme exponent, then suppose $\lambda > a$. According to the proof of Lemma 2.2, there exists an arbitrary small $\varepsilon > 0$ such that $\lambda \ominus \varepsilon > a \oplus \varepsilon$.

Thus, we have

$$\frac{|f(t)|}{e_{a\oplus\varepsilon}(t,t_0)} \geq \frac{|f(t)|}{e_{\lambda\ominus\varepsilon}(t,t_0)}, \text{ for all } t \in \mathbb{T}_{t_0}.$$

This implies a contradiction since

$$\lim_{t\to\infty} \frac{|f(t)|}{e_{a\oplus\varepsilon}(t,t_0)} = 0 \text{ and } \limsup_{t\to\infty} \frac{|f(t)|}{e_{\lambda\ominus\varepsilon}(t,t_0)} = \infty.$$

Hence, $\lambda \leq a$.

In case $\lambda = \infty$, $\lambda > a$ is obvious. If $\lambda < \infty$, then suppose $\lambda < a$. Repeating arguments as the above we also get a contradiction. Therefore $\lambda \geq a$.

v) Put $\alpha = \kappa_L[g]$. If $\alpha = \infty$ or α is the left extreme exponent then we have the proof. If $\alpha \in \mathbb{R} \cap \mathcal{R}^+$ then we get

$$\frac{|f(t)|}{e_{\alpha\oplus\varepsilon}(t,t_0)} \leq \frac{|g(t)|}{e_{\alpha\oplus\varepsilon}(t,t_0)} \text{ and } \lim_{t\to\infty} \frac{|g(t)|}{e_{\alpha\oplus\varepsilon}(t,t_0)} = 0.$$

Thus,

$$\lim_{t\to\infty} \frac{|f(t)|}{e_{\alpha\oplus\varepsilon}(t,t_0)} = 0.$$

According to Lemma 2.4.iv) we obtain $\kappa_L[f] \leq \alpha$.

vi) Assume that f is bounded above, there exists a number $M > 0$ such that $|f| \leq M$. The proof is implied from the inequality $\kappa_L[f] \leq \kappa_L[M] = 0$.

The proof is complete. □

We now set

$$\widehat{\Re}\lambda(t) := \lim_{s\searrow\mu(t)} \frac{|1+s\lambda|-1}{s} = \begin{cases} \Re\lambda & \text{if } \mu(t) = 0 \\ \dfrac{|1+\mu(t)\lambda|-1}{\mu(t)} & \text{if } \mu(t) \neq 0, \end{cases}$$

it follows that $\Re\lambda \leq \widehat{\Re}\lambda(t) \leq |\lambda|$, for all $t \in \mathbb{T}$, which deduces that

$$\Re\lambda \leq \liminf_{t\to\infty} \widehat{\Re}\lambda(t) \leq \limsup_{t\to\infty} \widehat{\Re}\lambda(t) \leq |\lambda|. \tag{2.5}$$

Lemma 2.5. *For any $\lambda \in \mathcal{R} \cap \mathbb{C}$, the following assertions hold true.*

i) $\kappa_L[e_\lambda(\cdot, t_0)] = \kappa_L[e_{\widehat{\Re}\lambda}(\cdot, t_0)]$;

ii) $\kappa_L[e_\lambda(\cdot, t_0)]$ *does not depend on* t_0;

iii) *If* $q(\cdot) \in \mathcal{R}^+$ *then*

$$\kappa_L[e_q(\cdot, t_0)] \leq \limsup_{t\to\infty} q(t); \tag{2.6}$$

iv) *We have*

$$\kappa_L[e_\lambda(\cdot, t_0)] \leq \limsup_{t\to\infty} \widehat{\Re}\lambda(t) \leq |\lambda|; \tag{2.7}$$

v) *We get*

$$\Re\lambda \leq \liminf_{t\to\infty} \widehat{\Re}\lambda(t) \leq \kappa_L[e_\lambda(\cdot, t_0)]. \tag{2.8}$$

Proof. i) It is known that $|e_\lambda(\cdot, t_0)| = e_{\widehat{\Re}\lambda}(\cdot, t_0)$ [36, Theorem 7.4]. Thus,

$$\kappa_L[e_\lambda(\cdot, t_0)] = \kappa_L[e_{\widehat{\Re}\lambda}(\cdot, t_0)].$$

ii) For $t_1 > t_0$, we have $e_\lambda(t, t_0) = e_\lambda(t, t_1)e_\lambda(t_1, t_0)$. Furthermore, since $\lambda \in \mathcal{R} \cap \mathbb{C}$, $e_\lambda(t_1, t_0) \neq 0$. Therefore, by Lemma 2.4.iii) we see that

$$\kappa_L[e_\lambda(\cdot, t_0)] = \kappa_L[e_\lambda(\cdot, t_1)].$$

iii) Set

$$\alpha := \limsup_{t\to\infty} q(t) = \lim_{T\to\infty} \sup_{t\geq T} q(t).$$

For any $\varepsilon > 0$, we can find an element $T_0 > t_0$ such that $q(t) \leq \alpha + \varepsilon$ for all $t \geq T_0$, which implies that $0 < e_{q(\cdot)}(t, T_0) \leq e_{\alpha+\varepsilon}(t, T_0)$. Hence, by Lemma 2.4.v), we have

$$\kappa_L\left[e_{q(\cdot)}(\cdot, T_0)\right] \leq \kappa_L\left[e_{\alpha+\varepsilon}(\cdot, T_0)\right] = \alpha + \varepsilon.$$

Since $\varepsilon > 0$ is arbitrarily small, $\kappa_L\left[e_\lambda(\cdot, t_0)\right] \leq \alpha$.

iv) This property is followed from (2.5) and Lemmas 2.5.i), 2.5.iii).

v) Set $\beta := \liminf_{t\to\infty} \widehat{\Re}\lambda(t) = \lim_{T\to\infty} \inf_{t\geq T} \widehat{\Re}\lambda(t)$. We see, by (2.5), that $\Re\lambda \leq \beta$. In case $\beta = -\frac{1}{\mu^*}$, the inequality is trivial, since $\kappa_L\left[e_\lambda(\cdot, t_0)\right] \geq -\frac{1}{\mu^*}$.

Next, we consider the case $\beta > -\frac{1}{\mu^*}$. For any number $\varepsilon > 0$ is sufficiently small, we can find an element $T_0 > t_0$ such that

$$-\frac{1}{\mu^*} < \beta - \varepsilon \leq \widehat{\Re}\lambda(t), \text{ for all } t \geq T_0.$$

Hence $0 < e_{\beta-\varepsilon}(\cdot, T_0) \leq e_{\widehat{\Re}\lambda}(\cdot, T_0)$, which implies that

$$\beta - \varepsilon = \kappa_L\left[e_{\beta-\varepsilon}(\cdot, T_0)\right] \leq \kappa_L\left[e_{\widehat{\Re}\lambda}(\cdot, T_0)\right] = \kappa_L\left[e_{\widehat{\Re}\lambda}(\cdot, t_0)\right] = \kappa_L\left[e_\lambda(\cdot, t_0)\right].$$

Thus, $\beta \leq \kappa_L\left[e_\lambda(\cdot, t_0)\right]$. The proof is complete. □

We have a remark that is deduced from Lemma 2.5.

Remark 2.6. There hold following statements.

i) If $\lambda \in \mathbb{R} \cap \mathcal{R}^+$, then $\widehat{\Re}\lambda(t) = \lambda$, and hence $\kappa_L[e_\lambda(\cdot, t_0)] = \lambda$;

ii) If $\mathbb{T} = \mathbb{R}$, then $\kappa_L[e_\lambda(\cdot, t_0)] = \chi[e^{\lambda(t-t_0)}] = \Re\lambda$ $(\lambda \in \mathbb{C})$;

iii) If $\mathbb{T}$ is a homogeneous time scale, i.e., $\mu(t) \equiv h \neq 0$, then

$$\kappa_L[e_\lambda(\cdot, t_0)] = \kappa_L[(1 + h\lambda)^{t-t_0}] = \frac{|1 + h\lambda| - 1}{h}.$$

Especially, if $\mathbb{T} = \mathbb{Z}$, then $\kappa_L[e_\lambda(\cdot, t_0)] = |1 + \lambda| - 1$.

Lemma 2.7. $\kappa_L[f + g] \leq \max\{\kappa_L[f], \kappa_L[g]\}$. *Furthermore, if* $\kappa_L[f] \neq \kappa_L[g]$, *then the equality holds.*

Proof. Set $\alpha := \kappa_L[f]$ and $\beta := \kappa_L[g]$, $\alpha, \beta \in \mathbb{R} \cap \mathcal{R}^+$. We consider the following two cases:

Case 1. If $\alpha \leq \beta$, then we deduce that

$$\frac{|(f+g)(t)|}{e_{\beta\oplus\varepsilon}(t, t_0)} \leq \frac{|f(t)|}{e_{\beta\oplus\varepsilon}(t, t_0)} + \frac{|g(t)|}{e_{\beta\oplus\varepsilon}(t, t_0)} \leq \frac{|f(t)|}{e_{\beta+\varepsilon}(t, t_0)} + \frac{|g(t)|}{e_{\beta+\varepsilon}(t, t_0)} \overset{t\to\infty}{\longrightarrow} 0,$$

and hence, $\kappa_L[f + g] \leq \beta$.

Case 2. If $\alpha < \beta$, we choose any number $\varepsilon > 0$ small enough such that $\alpha \oplus \varepsilon \leq \beta \ominus \varepsilon$ and then we get

$$\limsup_{t\to\infty} \frac{|(f+g)(t)|}{e_{\beta\ominus\varepsilon}(t,t_0)} \geq \limsup_{t\to\infty} \left(\frac{|g(t)|}{e_{\beta\ominus\varepsilon}(t,t_0)} - \frac{|f(t)|}{e_{\beta\ominus\varepsilon}(t,t_0)} \right)$$
$$\geq \limsup_{t\to\infty} \frac{|g(t)|}{e_{\beta\ominus\varepsilon}(t,t_0)} - \limsup_{t\to\infty} \frac{|f(t)|}{e_{\alpha\oplus\varepsilon}(t,t_0)} = \infty.$$

This means that $\kappa_L[f+g] \geq \beta$. The proof is complete. □

Remark 2.8. We have the following remarks:

i) If either α or β or both are the left extreme exponent or ∞, then the above inequality is also valid.

ii) We always have $\kappa_L\left[\sum_{i=1}^{n} c_i f_i\right] \leq \max_{1\leq i\leq n} \kappa_L[f_i]$, where f_i are continuous on $[t_0,\infty)_{\mathbb{T}}$, $c_i \neq 0$. Moreover, if there exists an index j such that $\kappa_L[f_j] > \kappa_L[f_i], j \neq i$, and $i,j = 1,2,...,n$, then $\kappa_L\left[\sum_{i=1}^{n} c_i f_i\right] = \kappa_L[f_j]$.

Since $\alpha,\beta \in \mathbb{R}\cap\mathcal{R}^+$, it is not sure to imply $\alpha+\beta \in \mathbb{R}\cap\mathcal{R}^+$, we cannot expect $\kappa_L[fg] \leq \kappa_L[f] + \kappa_L[g]$ as in the case $\mathbb{T} = \mathbb{R}$. However, we will have some similar results stated below.

Lemma 2.9. $\kappa_L[fg] \leq \kappa_L[e_{\kappa_L[f]\oplus\kappa_L[g]}(\cdot,t_0)]$.

Proof. Denote $\alpha = \kappa_L[f]$ and $\beta = \kappa_L[g]$, $\alpha,\beta \in \mathbb{R}\cap\mathcal{R}^+$. For any arbitrarily small number $\varepsilon > 0$, $t \in \mathbb{T}_0$, we have

$$\frac{|(fg)(t)|}{e_{\kappa_L[e_{\alpha\oplus\beta}(\cdot,t_0)]\oplus\varepsilon}(t,t_0)} = \frac{|f(t)|}{e_{\alpha\oplus\varepsilon_1}(t,t_0)} \cdot \frac{|g(t)|}{e_{\beta\oplus\varepsilon_2}(t,t_0)} \cdot \frac{e_{\alpha\oplus\beta}(t,t_0)e_{\varepsilon_1\oplus\varepsilon_2\oplus\varepsilon_3}(t,t_0)}{e_{\kappa_L[e_{\alpha\oplus\beta}(\cdot,t_0)]\oplus\varepsilon_3}(t,t_0)e_{\varepsilon}(t,t_0)},$$

where the positive numbers $\varepsilon_1, \varepsilon_2, \varepsilon_3$ are chosen such that $\varepsilon_1 \oplus \varepsilon_2 \oplus \varepsilon_3 \leq \varepsilon$, for all $t \in \mathbb{T}_{t_0}$. Since

$$\lim_{t\to\infty} \frac{|f(t)|}{e_{\alpha\oplus\varepsilon_1}(t,t_0)} = 0, \lim_{t\to\infty} \frac{|g(t)|}{e_{\alpha\oplus\varepsilon_2}(t,t_0)} = 0$$

and

$$\lim_{t\to\infty} \frac{e_{\alpha\oplus\beta}(t,t_0)}{e_{\kappa_L[e_{\alpha\oplus\beta}(\cdot,t_0)]\oplus\varepsilon_3}(t,t_0)} = 0,$$

it follows that

$$\lim_{t\to\infty} \frac{|(fg)(t)|}{e_{\kappa_L[e_{\alpha\oplus\beta}(\cdot,t_0)]\oplus\varepsilon}(t,t_0)} = 0.$$

According to Lemma 2.4.iv) we have $\kappa_L[fg] \leq \kappa_L[e_{\kappa_L[f]\oplus\kappa_L[g]}(\cdot,t_0)]$. The proof is complete. □

Definition 2.10. The function f is said to have exact Lyapunov exponent α, if for any $\varepsilon > 0$, $t \in \mathbb{T}_{t_0}$, we have

$$\lim_{t\to\infty} \frac{|f(t)|}{e_{\alpha\oplus\varepsilon}(t,t_0)} = 0 \quad \text{and} \quad \lim_{t\to\infty} \frac{|f(t)|}{e_{\alpha\ominus\varepsilon}(t,t_0)} = \infty.$$

Lemma 2.11. *If at least one of two functions, f or g, has exact Lyapunov exponent, then* $\kappa_L[fg] = \kappa_L[e_{\kappa_L[f]\oplus\kappa_L[g]}(\cdot,t_0)]$.

Proof. We assume that f has exact Lyapunov exponent. For any number $\varepsilon > 0$, there exists an increasing sequence $t_n \to \infty$ such that

$$\lim_{n\to\infty} \frac{|g(t_n)|}{e_{\kappa_L[g]\ominus\varepsilon}(t_n,t_0)} = \infty.$$

Since f has exact Lyapunov exponent, we conclude that

$$\lim_{n\to\infty} \frac{|f(t_n)|}{e_{\kappa_L[f]\ominus\varepsilon}(t_n,t_0)} = \infty.$$

Therefore,

$$\limsup_{t\to\infty} \frac{|(fg)(t)|}{e_{\kappa_L[f]\oplus\kappa_L[g]\ominus\varepsilon}(t,t_0)} \geq \lim_{n\to\infty} \frac{|f(t_n)|}{e_{\kappa_L[f]\ominus\frac{\varepsilon}{2}}(t_n,t_0)} \lim_{n\to\infty} \frac{|g(t_n)|}{e_{\kappa_L[g]\ominus\frac{\varepsilon}{2}}(t_n,t_0)} = \infty,$$

which implies that $\kappa_L[fg] \geq \kappa_L[e_{\kappa_L[f]\oplus\kappa_L[g]}(\cdot,t_0)]$. The proof is complete. □

Remark 2.12. In case that both functions, f and g, have exact Lyapunov exponents, then so does the function fg, and

$$\kappa_L[fg] = \kappa_L[e_{\kappa_L[f]\oplus\kappa_L[g]}(\cdot,t_0)].$$

In general, if all functions $f_1, f_2, ..., f_m$ have exact Lyapunov exponents then the function $f_1 f_2 \cdots f_m$ does, too, and

$$\kappa_L[f_1 f_2 \cdots f_m] = \kappa_L[e_{\kappa_L[f_1]\oplus\kappa_L[f_2]\oplus\cdots\oplus\kappa_L[f_m]}(\cdot,t_0)].$$

Remark 2.13. i) If $\mathbb{T} = \mathbb{R}$, $\kappa_L[fg] \leq \kappa_L[e_{\kappa_L[f] \oplus \kappa_L[g]}(\cdot, t_0)] = \kappa_L[f] + \kappa_L[g]$.

ii) If $\mathbb{T} = \mathbb{Z}$, $\kappa_L[fg] \leq \kappa_L[e_{\kappa_L[f] \oplus \kappa_L[g]}(\cdot, t_0)] = \kappa_L[f] + \kappa_L[g] + \kappa_L[f]\kappa_L[g]$ (or equivalently, $\chi[fg] \leq \chi[f] + \chi[g]$).

iii) Since $\kappa_L[f] \oplus \kappa_L[g](\cdot) \in \mathcal{R}^+$, due to the relation (2.6), we have that

$$\begin{aligned}
\kappa_L[fg] &\leq \limsup_{t\to\infty}\{\kappa_L[f] \oplus \kappa_L[g]\} \\
&= \limsup_{t\to\infty}\{(\kappa_L[f] + \kappa_L[g] + \mu(t)\kappa_L[f]\kappa_L[g])\} \\
&= \begin{cases} \kappa_L[f] + \kappa_L[g] + \kappa_L[f]\kappa_L[g] \limsup\limits_{t\to\infty} \mu(t), & \text{if } \kappa_L[f]\kappa_L[g] \geq 0 \\ \kappa_L[f] + \kappa_L[g] + \kappa_L[f]\kappa_L[g] \liminf\limits_{t\to\infty} \mu(t), & \text{if } \kappa_L[f]\kappa_L[g] < 0. \end{cases}
\end{aligned}$$

2.1.3 Lyapunov Exponent of Matrix Functions

The Lyapunov exponent of a matrix function

$$F(t) = [f_{ij}(t)]_{m\times n},\ i = 1, 2, ..., m,\ j = 1, 2, ..., n$$

is defined by

$$\kappa_L[F] := \max_{i,j} \kappa_L[f_{ij}],$$

where $f_{ij} : \mathbb{T}_{t_0} \to \mathbb{K}$ are the functions. It is clear that

$$\kappa_L[F] = \kappa_L[\|F\|]$$

and it satifies all Lemmas 2.4, 2.7, and 2.9, which are similar to the case of Lyapunov exponent for scalar functions.

2.1.4 Lyapunov Exponent of Integrals

Theorem 2.14. *Given a continuous function f defined on $\mathbb{T}_{t_0}$. Let*

$$F(t) = \begin{cases} \displaystyle\int_t^\infty f(\tau)\,\Delta\tau, & \text{if } \kappa_L[f] < 0 \\ \displaystyle\int_{t_0}^t f(\tau)\,\Delta\tau, & \text{if } \kappa_L[f] \geq 0. \end{cases}$$

Then, $\kappa_L[F] \leq \kappa_L[f]$.

Proof. Denote $\lambda = \kappa_L[f]$ and suppose that $\lambda \in \mathcal{R}^+$. By definition, for any $\varepsilon_1 > 0$ there exist $C > 0$ and $T_0 > t_0$ such that

$$|f(t)| \leq Ce_{\lambda \oplus \varepsilon_1}(t, t_0), \text{ for all } t \geq T_0. \tag{2.9}$$

Suppose that $\lambda < 0$. Let $\varepsilon > 0$ and choose positive numbers $\varepsilon_1, \varepsilon_2$, and ε_3 such that,

$$\lambda \oplus \varepsilon_1 \leq \lambda + \varepsilon_2 \leq \lambda \oplus \varepsilon \ominus \varepsilon_3, \text{ and } \lambda + \varepsilon_2 < 0,$$

for all $t > t_0$. We have

$$|F(t)| \leq C \int_t^\infty e_{\lambda \oplus \varepsilon_1}(\tau, t_0) \Delta\tau \leq C \int_t^\infty e_{\lambda + \varepsilon_2}(\tau, t_0)\, \Delta\tau = \frac{-C}{\lambda + \varepsilon_2} e_{\lambda + \varepsilon_2}(t, t_0),$$

which deduces that

$$|F(t)| \leq \frac{-C}{\lambda + \varepsilon_2} \frac{e_{\lambda \oplus \varepsilon}(t, t_0)}{e_{\varepsilon_3}(t, t_0)}. \tag{2.10}$$

for all $t \geq T_0$. Multiplying both sides of (2.10) by $e_{\ominus\lambda \oplus \varepsilon}(t, t_0)$, we get

$$\frac{|F(t)|}{e_{\lambda \oplus \varepsilon}(t, t_0)} \leq \frac{-C}{\lambda + \varepsilon_2} \frac{1}{e_{\varepsilon_3}(t, t_0)} \overset{t \to \infty}{\longrightarrow} 0, \text{ for all } \varepsilon > 0.$$

Using Lemma 2.4.iv), it follows that $\kappa_L[F] \leq \lambda$.

The case $\lambda \geq 0$ can be proven in a similar way. If $\lambda = \kappa_L[f]$ is the left extreme exponent or ∞, then we also have $\kappa_L[F] \leq \kappa_L[f]$. The proof is complete. □

2.2 Lyapunov Exponents of Solutions of Linear Equation

2.2.1 Lyapunov Spectrum of Linear Equation

Consider the linear equation

$$x^\Delta = A(t)x, \tag{2.11}$$

where $A(t)$ is a regressive and rd-continuous $n \times n$-matrix on time scale $\mathbb{T}$. It is known that Equation (2.11) with the initial value $x(t_0) = x_0$ has a unique solution $x(t) = x(t; t_0, x_0)$ on $\mathbb{T}$.

Theorem 2.15. *Let $\mathcal{M} = \limsup_{t\to\infty} \|A(t)\|$. If $x(\cdot)$ is a nontrivial solution of Equation* (2.11)*, then $\kappa_L[x(\cdot)] \leq \mathcal{M}$. Furthermore, if $\limsup_{t\to\infty} \mu(t) < \frac{1}{\mathcal{M}}$, then the appreciation $-\mathcal{M} \leq \kappa_L[x(\cdot)] \leq \mathcal{M}$ holds.*

Proof. Since $\mathcal{M} = \limsup_{t\to\infty} \|A(t)\|$, it deduces that $\|A(t)\| \leq \mathcal{M}$, for all $t \in \mathbb{T}$. Let $x = (x_1, \ldots, x_n)^T \neq 0$ and $t_0, t \in \mathbb{T}$. By the assumption, we have

$$x(t) = x(t_0) + \int_{t_0}^{t} A(\tau)x(\tau)\Delta\tau.$$

Hence

$$\|x(t)\| \leq \|x(t_0)\| + \int_{t_0}^{t} \|A(\tau)\|\|x(\tau)\|\Delta\tau.$$

By applying Gronwall's inequality for all $t \geq t_0$, we obtain

$$\|x(t)\| \leq \|x(t_0)\| e_{\|A(t)\|}(t, t_0),$$

or

$$\frac{\|x(t)\|}{\|x(t_0)\|} \leq e_{\|A(t)\|}(t, t_0) \leq e_{\mathcal{M}}(t, t_0).$$

Hence, we have

$$\kappa_L[x(\cdot)] \leq \kappa_L[e_{\mathcal{M}}(t, t_0)] = \mathcal{M}.$$

We now prove the second assertion. Let $T_1 > t_0$, such that $\mu(t) < \frac{1}{\mathcal{M}}$ is satisfied for all $t \geq T_1$. We can directly see that $\Phi_A^{-1}(t, T_1)$ solves the adjoint dynamic equation

$$[\Phi_A^{-1}(t, T_1)]^{\Delta} = -\Phi_A^{-1}(\sigma(t), T_1)A(t) = -\Phi_A^{-1}(t, T_1)(I + \mu(t)A(t))^{-1}A(t).$$

Therefore,

$$\Phi_A^{-1}(t, T_1) = I - \int_{T_1}^{t} \Phi_A^{-1}(\tau, T_1)[I + \mu(\tau)A(\tau)]^{-1}A(\tau)\,\Delta\tau.$$

Hence,

$$\|\Phi_A^{-1}(t, T_1)\| \leq 1 + \int_{T_1}^{t} \|(I + \mu(\tau)A(\tau))^{-1}\|\|A(\tau)\|\|\Phi_A^{-1}(\tau, T_1)\|\,\Delta\tau.$$

Continuing to apply Gronwall's inequality for the above inequation gets

$$\|\Phi_A^{-1}(t, T_1)\| \leq e_{\|(I+\mu(t)A(t))^{-1}\|\|A(t)\|}(t, T_1),$$

which implies that

$$\|\Phi_A^{-1}(t, T_1)\|^{-1} \geq e_{\ominus\|(I+\mu(t)A(t))^{-1}\|\|A(t)\|}(t, T_1). \tag{2.12}$$

Since $\mu(t)\|A(t)\| < 1$, by Hille-Yosida's Theorem, see [13, Theorem 7.4], we have

$$\|(I + \mu(t)A(t))^{-1}\| \leq \frac{1}{1 - \mu(t)\|A(t)\|}.$$

This deduces

$$\ominus\|(I+\mu(t)A(t))^{-1}\|\|A(t)\| \geq -\|A(t)\| \geq -\mathcal{M}, \tag{2.13}$$

for all $t \geq T_1$. Furthermore,

$$\frac{\|x(t)\|}{\|x(T_1)\|} \geq \|\Phi_A^{-1}(t,T_1)\|^{-1} \tag{2.14}$$

Due to (2.12), (2.13), (2.14) and properties of exponential function on time scale, we get

$$\kappa_L[x(\cdot)] \geq \kappa_L[e_{-\mathcal{M}}(t,T_1)] = -\mathcal{M}.$$

The proof is complete. □

Remark 2.16. If $\mathbb{T} = \mathbb{R}$, then $\mu(t) \equiv 0$ and we get a popular inequality

$$-\mathcal{M} \leq \kappa_L[x(\cdot)] = \chi[x(\cdot)] \leq \mathcal{M}.$$

We now define the Lyapunov spectrum of Equation (2.11).

Definition 2.17. The set of all normal Lyapunov exponents of solutions of Equation (2.11) is called the *Lyapunov spectrum* of this equation.

Theorem 2.18. *The Lyapunov spectrum of Equation* (2.11) *has n distinct values at most.*

Proof. Firstly, we denote $x_1, ..., x_m$, for $1 \leq m \leq n$ the solutions of Equation (2.11). By Lemma 2.7, if $\kappa_L[x_i] \neq \kappa_L[x_j]$ for $i \neq j$, then

$$\kappa_L[x_i + x_j] = \max\{\kappa_L[x_i], \kappa_L[x_j]\}$$

and

$$\kappa_L[k_1x_1 + \cdots + k_mx_m] \leq \max\{\kappa_L[x_i], 1 \leq i \leq m\}.$$

Next, we will prove that, if $x_i \neq 0$, for all $1 \leq i \leq m$, and the numbers $\kappa_L[x_i], 1 \leq i \leq m$, are distinct then $x_1, \ldots, x_m$ are linearly independent. Indeed, on the contrary, suppose that the solutions $x_1, \ldots, x_m$ are linearly dependent, i.e., there exist the constants $k_1, \ldots, k_m$ are not simultaneously equal to zero at all, such that

$$k_1x_1 + \cdots + k_mx_m = 0,$$

while, $\kappa_L[x_1], \ldots, \kappa_L[x_m]$ are distinct. On the other hand, by above proofs and properties of Lyapunov exponent, we have

$$-\infty = \kappa_L[k_1x_1 + \cdots + k_mx_m] = \max\{\kappa_L[x_i], x_i \neq 0, 1 \leq i \leq m\} \neq -\infty.$$

This contradiction implies that the system $x_1, x_2, ..., x_m$ is linearly independent. The proof is complete. □

2.2.2 Lyapunov Inequality

Assume that $X(t,t_0)$ is the basic solution matrix of Equation (2.11), and satisfies the condition $X(t_0,t_0) = X_0 \in \mathbb{R}^{n\times n}$. Set $W(t,t_0) := \det(X(t,t_0))$. It is clear that W is a solution of the equation $W^\Delta = \alpha(t)W$ (see [47]), where $\alpha(t)$ is defined as follows:

$$\alpha(t) := \lim_{s\searrow\mu(t)} \frac{\det(I + sA(t)) - 1}{s} = \begin{cases} \operatorname{trace} A(t) & \text{if } \mu(t) = 0 \\ \dfrac{\det(I + \mu(t)A(t)) - 1}{\mu(t)} & \text{if } \mu(t) \neq 0. \end{cases}$$

Since the matrix function $A(\cdot) \in C_{\mathrm{rd}}\mathcal{R}(\mathbb{T}, \mathbb{K}^{\mathrm{n}\times\mathrm{n}})$, the graininess function $\mu(t) = \sigma(t) - t$ are rd-continuous, and $\alpha(\cdot) \in C_{\mathrm{rd}}\mathcal{R}(\mathbb{T}, \mathbb{C})$, the system

$$\begin{cases} W^\Delta = \alpha(t)W \\ W(t_0,t_0) = \det(X_0) \end{cases}$$

has a unique solution $W(t,t_0) = \det(X_0)e_\alpha(t,t_0)$, for all $t \in \mathbb{T}$.

Let $\{x_1(t), x_2(t), ..., x_n(t)\}$ be a system of regular fundamental solutions of Equation (2.11), i.e., the system of these solutions has a property: The Lyapunov exponent of solutions combined from some arbitrary solutions of this system will be equal to the Lyapunov exponent of a solution attending in the combination. In other words, if

$$x(t) = k_1x_1(t) + k_2x_2(t) + \cdots + k_nx_n(t),$$

then

$$\kappa_L[x(\cdot)] = \kappa_L[x_i(\cdot)]$$

with some $i \in \{1, \ldots, n\}$ (by the finiteness of the Lyapunov spectrums' set, we can find such a fundamental solution system).

Denote by

$$S = \{\alpha_1, \alpha_2, \dots, \alpha_n | \alpha_1 \le \alpha_2 \le \cdots \le \alpha_n\}$$

the set of Lyapunov spectrum of Equation (2.11). In addition, we suppose that $\alpha_i \in \mathbb{R} \cap \mathcal{R}^+$, for all $i = 1, 2, ..., n$.

Theorem 2.19 (Lyapunov's Inequality). $\kappa_L[e_\alpha(\cdot, t_0)] \le \kappa_L[e_{\alpha_1 \oplus \alpha_2 \oplus \ldots \oplus \alpha_n}(\cdot, t_0)]$.

Proof. By definition we have

$$W = \sum_{\sigma \in \Theta} \operatorname{sign}(\sigma)\, x_{\sigma(1)1} \dots x_{\sigma(n)n},$$

where $x_i = (x_{1i}, x_{2i}, \dots, x_{ni})^T$ and Θ is the set of all permutations of n elements $1, 2, \dots, n$. Therefore,

$$\begin{aligned}
\kappa_L[W] &\le \max_{\sigma \in \Theta} \kappa_L \left[x_{\sigma(1)1} \dots x_{\sigma(n)n} \right] \\
&\le \max_{\sigma \in \Theta} \kappa_L \left[e_{\kappa_L[x_{\sigma(1)1}] \oplus \cdots \oplus \kappa_L[x_{\sigma(n)n}]}(\cdot, t_0) \right] \\
&= \max_{\sigma \in \Theta} \kappa_L \left[e_{\kappa_L[x_{\sigma(1)1}]}(\cdot, t_0) \cdots e_{\kappa_L[x_{\sigma(n)n}]}(\cdot, t_0) \right] \\
&\le \kappa_L \left[e_{\alpha_1}(\cdot, t_0) \cdots e_{\alpha_n}(\cdot, t_0) \right] \\
&= \kappa_L \left[e_{\alpha_1 \oplus \cdots \oplus \alpha_n}(\cdot, t_0) \right].
\end{aligned}$$

Thus we get

$$\kappa_L[e_\alpha(\cdot, t_0)] \le \kappa_L[e_{\alpha_1 \oplus \cdots \oplus \alpha_n}(\cdot, t_0)].$$

The proof is complete. □

Remark 2.20. In case $\mathbb{T} = \mathbb{R}$, we have

$$\kappa_L[e_\alpha(\cdot, t_0)] = \limsup_{t \to \infty} \frac{1}{t - t_0} \int_{t_0}^{t} (\operatorname{trace} A(s)) ds,$$

and

$$\kappa_L \left[e_{\alpha_1 \oplus \cdots \oplus \alpha_n}(\cdot, t_0) \right] = \alpha_1 + \cdots + \alpha_n.$$

Thus, we get the Lyapunov inequality for the ordinary differential equations in [56].

Remark 2.21. The question of whether equality $\kappa_L[e_\alpha(\cdot, t_0)] = \kappa_L[e_{\oplus_{i=1}^n \alpha_i}(\cdot, t_0)]$ is true or not is still an open problem even if matrix A is constant. A partial answer will be obtained if we have one more condition.

We consider Equation (2.11), where $A(t) \equiv A$ is a constant and regressive $n \times n$-matrix. Let $\lambda_i, i = 1, 2, ..., n$ be the eigenvalues of matrix A. We will show that

$$\alpha(t) = \lambda_1 \oplus \lambda_2 ... \oplus \lambda_n, \tag{2.15}$$

where the function on the right-hand side is introduced in Definition 1.3. Indeed, without loss of generality, we suppose that

$$\det(A - \lambda I) = (-1)^n \lambda^n + (-1)^{n-1} a_{n-1} \lambda^{n-1} + \cdots - a_1 \lambda + a_0.$$

Then, by the Viète theorem

$$\sum_{1 \le i_1 < i_2 < \cdots < i_k \le n} \lambda_{i_1} \lambda_{i_2} \cdots \lambda_{i_k} = a_{n-k},$$

for all $k = 1, 2, ..., n$. Therefore,

$$\begin{aligned}\alpha(t) &= \lim_{s \searrow \mu(t)} \frac{\det(I + sA) - 1}{s} \\ &= a_0 \mu^{n-1}(t) + a_1 \mu^{n-2}(t) + \cdots + a_{n-2} \mu(t) + a_{n-1}.\end{aligned}$$

On the other hand, by induction, we get

$$\begin{aligned}\lambda_1 \oplus \lambda_2 ... \oplus \lambda_n &= \sum \lambda_i + \sum_{i<j} \lambda_i \lambda_j \mu(t) \\ &\quad + \sum_{i<j<k} \lambda_i \lambda_j \lambda_k \mu^2(t) + \cdots + \lambda_1 \cdots \lambda_n \mu^{n-1}(t) \\ &= a_0 \mu^{n-1}(t) + a_1 \mu^{n-2}(t) + \cdots + a_{n-1} = \alpha(t).\end{aligned}$$

Hence, we get the following theorem,

Theorem 2.22. *If for any eigenvalue λ_i of matrix A, the exponential function $e_{\lambda_i}(\cdot, t_0)$ has exact Lyapunov exponent, then $\kappa_L[e_\alpha(\cdot, t_0)] = \kappa_L[e_{\alpha_1 \oplus \alpha_2 \oplus ... \oplus \alpha_n}(\cdot, t_0)]$, where $\alpha_i = \kappa_L[e_{\lambda_i}(\cdot, t_0)]$, $i = 1, 2, ..., n$.*

Proof. From Remark 2.12 and equality (2.15) we have

$$\begin{aligned}\kappa_L[e_\alpha(\cdot, t_0)] &= \kappa_L[e_{\oplus_{i=1}^n \lambda_i}(\cdot, t_0)] = \kappa_L[\Pi_{i=1}^n e_{\lambda_i}(\cdot, t_0)] \\ &= \kappa_L[e_{\oplus_{i=1}^n \kappa_L[e_{\lambda_i}(\cdot, t_0)]}(\cdot, t_0)] = \kappa_L[e_{\oplus_{i=1}^n \alpha_i}(\cdot, t_0)].\end{aligned}$$

The proof is complete. □

2.3 Lyapunov Spectrum and Stability of Linear Equations

Consider the equation

$$x^{\Delta} = A(t)x, \tag{2.16}$$

where $A(t)$ is a regressive, rd-continuous $n \times n$-matrix, and $\|A(t)\| \leq \mathcal{M}$, for all $t \in \mathbb{T}_{t_0}$.

Definition 2.23. The trivial solution $x(t) \equiv 0$ of Equation (2.16) is said to be *exponentially asymptotically stable* if all solutions $x(t)$ of Equation (2.16) with the initial value $x(t_0)$ satisfy the relation

$$\|x(t)\| \leq N\|x(t_0)\|e_{-\alpha}(t, t_0), t \in \mathbb{T}_{t_0},$$

for some positive constants $N = N(t_0)$ and $\alpha > 0$ with $-\alpha \in \mathcal{R}^+$.

If the constant N can be chosen to be independent of t_0, then this solution is called *uniformly exponentially asymptotically stable.*

Theorem 2.24. *Consider Equation* (2.16) *with the stated conditions on* $A(\cdot)$. *Then,*

i) *Equation* (2.16) *is exponentially asymptotically stable if and only if there exists a constant* $\alpha > 0$ *with* $-\alpha \in \mathcal{R}^+$ *such that for every* $t_0 \in \mathbb{T}$, *there is a number* $N = N(t_0) \geq 1$ *such that*

$$\|\Phi_A(t, t_0)\| \leq Ne_{-\alpha}(t, t_0) \text{ for all } t \in \mathbb{T}_{t_0}.$$

ii) *Equation* (2.16) *is uniformly exponentially asymptotically stable if and only if there exist constants* $\alpha > 0,\ N \geq 1$ *with* $-\alpha \in \mathcal{R}^+$ *such that*

$$\|\Phi_A(t, t_0)\| \leq Ne_{-\alpha}(t, t_0) \text{ for all } t \in \mathbb{T}_{t_0}.$$

Proof. Every solution of Equation (2.16) satisfying the initial condition $x(t_0) = x_0$ can be expressed by $x(t) = \Phi_A(t, t_0)x_0$. Combining with the definition of exponential stability we have the proof. □

In the following theorem we give the spectral condition for exponential stability.

Theorem 2.25. *Let* $-\alpha := \max S$, *where* S *is the set of Lyapunov spectra of Equation* (2.16). *Then, Equation* (2.16) *is exponentially asymptotically stable if and only if* $\alpha > 0$.

Proof. The proof is divided into two parts.

Necessity. Suppose that Equation (2.16) is exponentially asymptotically stable. Then, there exist numbers $N \geq 1$, and $\alpha_1 > 0$, $-\alpha_1 \in \mathcal{R}^+$ such that

$$\|x(t)\| \leq N e_{-\alpha_1}(t, t_0)$$

for any solution $x(t)$ of Equation (2.16). By Lemma 2.4.v) we have

$$\kappa_L[x(\cdot)] \leq -\alpha_1.$$

This means that $-\alpha = \max S \leq -\alpha_1 < 0$.

Sufficiency. Suppose that $\alpha > 0$, and let

$$\{x_i(\cdot) = (x_{1i}(\cdot), x_{2i}(\cdot), \ldots, x_{ni}(\cdot))^T\}, i = 1, 2, ..., n$$

be a system of fundamental solutions of Equation (2.16). Hence, we have

$$\kappa_L[x_i(\cdot)] \leq -\alpha < 0$$

for all $i = 1, 2, ..., n$, which implies that

$$\lim_{t \to \infty} \frac{\|x_i(t)\|}{e_{-\frac{\alpha}{2}}(t, t_0)} = 0.$$

Therefore, there exists a number $T_0 > t_0$, such that

$$\|x_i(t)\| \leq e_{-\frac{\alpha}{2}}(t, t_0), \text{ for all } t \in \mathbb{T}, t_0 \leq t \leq T_0, \ i = 1, 2, ..., n.$$

We choose a number $N^* \geq 1$, such that

$$N^* \geq \sup_{1 \leq i \leq n, t_0 \leq t \leq T_0} \frac{\|x_i(t)\|}{e_{-\frac{\alpha}{2}}(t, t_0)}$$

and then obtain

$$\sup_{1 \leq i \leq n, t_0 \leq t \leq T_0} \|x_i(t)\| \leq N^* e_{-\frac{\alpha}{2}}(t, t_0).$$

If $x(\cdot)$ is an arbitrary nontrivial solution of Equation (2.16), then there are constants $a_1, a_2, ..., a_n$, such that

$$x(t) = \sum_{i=1}^{n} a_i x_i(t).$$

Since $\{x_1(t_0), x_2(t_0), ..., x_n(t_0)\}$ forms a basic of $\mathbb{R}^n$ and the norms are equivalent to each other in $\mathbb{R}^n$, there is a positive constant K, independent of $x(t_0)$, such that

$$K\|x(t_0)\| \geq \sum_{i=1}^{n} |a_i|.$$

Hence,

$$\|x(t)\| \leq \sum_{i=1}^{n} |a_i| \|x_i(t)\| \leq N^* \left(\sum_{i=1}^{n} |a_i| \right) e_{-\frac{\alpha}{2}}(t, t_0) \leq N\|x(t_0)\| e_{-\frac{\alpha}{2}}(t, t_0),$$

where $N := KN^*$. This means that Equation (2.16) is exponentially asymptotically stable. The proof is complete. □

We now consider the following equation

$$x^{\Delta} = Ax, \tag{2.17}$$

where A is a regressive constant matrix. Denote the set of all eigenvalues of matrix A by $\sigma(A)$. From the regressivity of A, it follows that $\sigma(A) \subset \mathcal{R}$.

Theorem 2.26. i) *If Equation* (2.17) *is exponentially asymptotically stable, then*

$$\kappa_L[e_\lambda(\cdot, t_0)] < 0, \text{ for all } \lambda \in \sigma(A).$$

ii) *Suppose that all eigenvalues of A are uniformly regressive. Then, the assumption $\kappa_L[e_\lambda(\cdot, t_0)] < 0$ implies that Equation* (2.17) *is exponentially asymptotically stable.*

Proof. Suppose that Equation (2.17) is exponentially asymptotically stable. Let $\lambda \in \sigma(A)$ and x_0 be its corresponding eigenvector. Since

$$x(t; t_0, x_0) = e_\lambda(t, t_0) x_0$$

is a solution of (2.17), we have $e_\lambda(t, t_0)\|x_0\| = \|x(t; t_0, x_0)\| \leq N\|x_0\| e_{-\alpha}(t, t_0)$, where $N \geq 1, \alpha > 0, -\alpha \in \mathcal{R}^+$. Hence,

$$\kappa_L[e_\lambda(\cdot, t_0)] \leq -\alpha < 0.$$

Next, to prove the second assertion, we define the sequence of λ-polynomials by

$$p_0^\lambda(t, s) := 1, \quad p_k^\lambda(t, s) := \int_s^t \frac{1}{1 + \lambda\mu(\tau)} p_{k-1}^\lambda(\tau, s) \Delta\tau.$$

By using this notation, we get an explicit representation for the exponential matrix function on time scale (see [16])

$$\Phi_A(t,t_0) = \sum_{i=1}^{m} \sum_{k=1}^{s_i} R_{ik} p_{k-1}^{\lambda_i}(t,t_0) e_{\lambda_i}(t,t_0), \tag{2.18}$$

where R_{ik} are the constants, and $\lambda_1, \lambda_2, ..., \lambda_m$ are distinct eigenvalues of the matrix A with respective multiples $s_1, s_2, ..., s_m,\ m \le n$.

We assume that λ is uniformly regressive, and $\kappa_L[e_\lambda(\cdot,t_0)] < 0$ for every $\lambda \in \sigma(A)$.

Let $\varepsilon > 0$ be an arbitrarily small number. By using L'Hôpital's rule, we get

$$\begin{aligned}
\lim_{t\to\infty} \frac{|p_1^\lambda(t,t_0)|}{e_\varepsilon(t,t_0)} &\le \lim_{t\to\infty} \frac{\int_{t_0}^{t} \frac{1}{|1+\lambda\mu(\tau)|}\Delta\tau}{e_\varepsilon(t,t_0)} \\
&= \lim_{t\to\infty} \frac{\left(\int_{t_0}^{t} \frac{1}{|1+\lambda\mu(\tau)|}\Delta\tau\right)^\Delta}{(e_\varepsilon(t,t_0))^\Delta} \\
&= \lim_{t\to\infty} \frac{1}{\varepsilon|1+\lambda\mu(t)|e_\varepsilon(t,t_0)} \\
&\le \lim_{t\to\infty} \frac{1}{\varepsilon\delta e_\varepsilon(t,t_0)} = 0.
\end{aligned}$$

Since ε is arbitrarily small, it follows from Lemma 2.4.iv) that $\kappa_L[p_1^\lambda(\cdot,t_0)] \le 0$. By induction, we get $\kappa_L[p_k^\lambda(\cdot,t_0)] \le 0,\ k = 1,2,...,s_i$ and $i = 1,2,...,m$. Therefore,

$$\begin{aligned}
\kappa_L[p_k^\lambda(t,t_0)e_\lambda(t,t_0)] &\le \kappa_L[e_{\kappa_L[p_k^\lambda(t,t_0)]\oplus\kappa_L[e_\lambda(t,t_0)]}(t,t_0)] \\
&= \kappa_L[e_{\kappa_L[p_k^\lambda(t,t_0)]}(t,t_0)e_{\kappa_L[e_\lambda(t,t_0)]}(t,t_0)] \\
&\le \kappa_L[e_{\kappa_L[e_\lambda(t,t_0)]}(t,t_0)] = \kappa_L[e_\lambda(t,t_0)] \\
&< 0.
\end{aligned}$$

Combining the inequality $\kappa_L[p_k^\lambda(t,t_0)e_\lambda(t,t_0)] < 0$ with the expression (2.18) and Theorem 2.25 obtains the proof. □

Corollary 2.27. *If for any eigenvalue $\lambda \in \sigma(A)$ we have $\Im\lambda \neq 0$ and $\kappa_L[e_\lambda(\cdot,t_0)] < 0$, then Equation (2.17) is exponentially asymptotically stable.*

Proof. The proof follows from the fact that if $\Im\lambda \neq 0$ then λ is uniformly regressive. □

Theorem 2.28. *Suppose that* $\limsup_{t\to\infty} \widehat{\Re}\lambda(t) < 0$, *for all* $\lambda \in \sigma(A)$. *Then, Equation* (2.17) *is exponentially asymptotically stable.*

Proof. From the assumption and the inequality (2.7), we see that

$$\kappa_L[e_\lambda(\cdot, t_0)] < 0, \text{ for all } \lambda \in \sigma(A).$$

Set

$$\alpha := \limsup_{t\to\infty} \widehat{\Re}\lambda(t) < 0, \lambda \in \sigma(A).$$

Choose $0 < \varepsilon \leq -\frac{\alpha}{2}$. Then, there exists an element $T_0 \in \mathbb{T}$ such that $\sup_{t\geq T_0} \widehat{\Re}\lambda(t) \leq \alpha + \varepsilon$, which implies that

$$(\widehat{\Re}\lambda \oplus \varepsilon)(t) \leq \frac{\alpha}{2} < 0, \text{ for all } t \geq T_0.$$

Hence, $\lim_{t\to\infty} e_{\widehat{\Re}\lambda\oplus\varepsilon}(t, t_0) = 0$. By applying L'Hôpital's rule, we obtain

$$\begin{aligned}
\limsup_{t\to\infty} |p_1^\lambda(t, t_0) e_{\lambda\oplus\varepsilon}(t, t_0)| &\leq \limsup_{t\to\infty} \int_{t_0}^t \frac{1}{|1 + \lambda\mu(\tau)|} \Delta\tau \cdot e_{\widehat{\Re}\lambda\oplus\varepsilon}(t, t_0) \\
&= \lim_{t\to\infty} \frac{\left(\int_{t_0}^t \frac{1}{|1+\lambda\mu(\tau)|}\Delta\tau\right)^\Delta}{\left(e_{\ominus(\widehat{\Re}\lambda\oplus\varepsilon)}(t, t_0)\right)^\Delta} \\
&= \lim_{t\to\infty} \frac{e_{\widehat{\Re}\lambda\oplus\varepsilon}(t, t_0)}{\ominus(\widehat{\Re}\lambda \oplus \varepsilon)(t)|1 + \lambda\mu(t)|}.
\end{aligned}$$

Since

$$\begin{aligned}
\frac{e_{\widehat{\Re}\lambda\oplus\varepsilon}(t, t_0)}{\ominus(\widehat{\Re}\lambda \oplus \varepsilon)(t)|1 + \lambda\mu(t)|} &= \frac{(1 + \varepsilon\mu(t) + \mu(t)\widehat{\Re}\lambda(t)(1 - \varepsilon\mu(t)) e_{\widehat{\Re}\lambda\oplus\varepsilon}(t, t_0)}{-(\widehat{\Re}\lambda \oplus \varepsilon)(t)|1 + \lambda\mu(t)|} \\
&\leq \frac{(1 + \varepsilon\mu(t) + \mu(t)\widehat{\Re}\lambda(t)(1 + \varepsilon\mu(t)) e_{\widehat{\Re}\lambda\oplus\varepsilon}(t, t_0)}{-(\widehat{\Re}\lambda \oplus \varepsilon)(t)|1 + \lambda\mu(t)|} \\
&= \frac{(1 + \varepsilon\mu(t))(1 + \mu(t)\widehat{\Re}\lambda(t)) e_{\widehat{\Re}\lambda\oplus\varepsilon}(t, t_0)}{-(\widehat{\Re}\lambda \oplus \varepsilon)(t)|1 + \lambda\mu(t)|} \\
&\leq \frac{(1 + \varepsilon\mu(t))|1 + \lambda\mu(t)| e_{\widehat{\Re}\lambda\oplus\varepsilon}(t, t_0)}{-(\widehat{\Re}\lambda \oplus \varepsilon)(t)|1 + \lambda\mu(t)|},
\end{aligned}$$

we get

$$\limsup_{t\to\infty} |p_1^\lambda(t, t_0) e_{\lambda\oplus\varepsilon}(t, t_0)| \leq \lim_{t\to\infty} \frac{(1 + \varepsilon\mu(t)) e_{\widehat{\Re}\lambda\oplus\varepsilon}(t, t_0)}{-(\widehat{\Re}\lambda \oplus \varepsilon)(t)} = 0.$$

Therefore, $p_1^\lambda(t,t_0)e_{\lambda\oplus\varepsilon}(t,t_0)$ is bounded from above by a certain constant C when t is large enough, which implies that

$$|p_1^\lambda(t,t_0)e_\lambda(t,t_0)| = |p_1^\lambda(t,t_0)e_{\lambda\oplus\varepsilon}(t,t_0)|e_{\ominus\varepsilon}(t,t_0) \le Ce_{\ominus\varepsilon}(t,t_0).$$

Thus,

$$\begin{aligned}\kappa_L[p_1^\lambda(t,t_0)e_\lambda(t,t_0)] &\le \kappa_L[Ce_{\ominus\varepsilon}(t,t_0)] \\ &\le \sup_{t\in\mathbb{T}}(\ominus\varepsilon) = \sup_{t\in\mathbb{T}}\left(-\frac{\varepsilon}{1+\varepsilon\mu(t)}\right) \\ &\le -\frac{\varepsilon}{1+\varepsilon\mu^*} < 0.\end{aligned}$$

By induction, we can prove that

$$\kappa_L[p_k^\lambda(t,t_0)e_\lambda(t,t_0)] < 0, \text{ for all } k = 0,1,2,...$$

We use expression (2.18), Theorem 2.25 and complete the proof. □

Note that if $\lambda(\cdot) \in \mathcal{R}^+$, then $\widehat{\Re}\lambda(t) = \lambda(t)$ for all $t \in \mathbb{T}$. Therefore, we get a corollary of Theorem 2.28.

Corollary 2.29. *If $\sigma(A) \subset (-\infty, 0) \cap \mathcal{R}^+$ then Equation* (2.17) *is exponentially asymptotically stable.*

To end Chapter 2, we consider an example.

Example 2.30. Considering Equation $x^\Delta(t) = Ax(t)$ on time scale

$$\mathbb{T} = \cup_{k=0}^{\infty}[2k, 2k+1],$$

with

$$A = \frac{1}{24}\begin{pmatrix} -24 & 0 & 48 \\ 1 & -24 & 24 \\ 33 & -72 & -48 \end{pmatrix}.$$

It is clear that

$$\mu(t) = \begin{cases} 0 & \text{if } t \in \cup_{k=0}^{\infty}[2k, 2k+1), \\ 1 & \text{if } t \in \cup_{k=0}^{\infty}\{2k+1\}, \end{cases}$$

the left extreme exponent is -1. Further,

$$\sigma(A) = \left\{-2, -1+\frac{1}{2}i, -1-\frac{1}{2}i\right\},$$

and all $\lambda \in \sigma(A)$ are uniformly regressive. We consider the following cases:

i) In case $\lambda_1 = -2$ and $t \in [2k, 2k+1]$, we have

$$\begin{aligned} e_{-2}(t,0) &= \exp\left(-\int_0^t 2\Delta s\right) \Pi_{\tau \in I_{0,t}} (1 - 2\mu(\tau)) \exp\left(\int_\tau^{\sigma(\tau)} 2\Delta s\right) \\ &= e^{-2t}(-1)^k e^{2k} = (-1)^k e^{-2(t-k)}. \end{aligned}$$

On the other hand, for all $t \in [2k, 2k+1]$,

$$\begin{aligned} e_{-\frac{1}{2}}(t,0) &= \exp\left(-\int_0^t \frac{1}{2}\Delta s\right) \Pi_{\tau \in I_{0,t}} \left(1 - \frac{1}{2}\mu(\tau)\right) \exp\left(\int_\tau^{\sigma(\tau)} \frac{1}{2}\Delta s\right) \\ &= e^{-\frac{1}{2}t} \frac{1}{2^k} e^{\frac{1}{2}k} = \frac{1}{2^k} e^{-\frac{1}{2}(t-k)}. \end{aligned}$$

By comparing these expressions, we see that there exists $c > 0$ such that

$$e_{-2}(t,0) \le c e_{-\frac{1}{2}}(t,0).$$

Hence

$$\kappa_L[e_{-2}(\cdot,0)] \le \kappa_L[e_{-\frac{1}{2}}(\cdot,0)] = -\frac{1}{2} < 0.$$

ii) In case $\lambda_2 = -1 + \frac{i}{2}$, we have

$$\widehat{\Re}\lambda_2(t) = \lim_{s \searrow \mu(t)} \frac{|1 + s(-1 + \frac{i}{2})| - 1}{s} = \begin{cases} -1 & \text{if } \mu(t) = 0, \\ \frac{1}{\sqrt{2}} - 1 & \text{if } \mu(t) = 1, \end{cases}$$

thus

$$\kappa_L[e_{\lambda_2}(\cdot,0)] \le \limsup_{t \to \infty} \widehat{\Re}\lambda_2(t) = \frac{1}{\sqrt{2}} - 1 < 0.$$

iii) Similarly, in case $\lambda_3 = -1 - \frac{i}{2}$, we also get

$$\kappa_L[e_{\lambda_3}(\cdot,0)] \le \limsup_{t \to \infty} \widehat{\Re}\lambda_3(t) = \frac{1}{\sqrt{2}} - 1 < 0.$$

Therefore, by Theorem 2.26, the above equation is exponentially asymptotically stable.

Make a note that Equation $x^\Delta(t) = -2x(t)$, $t \in \mathbb{T} = \cup_{k=0}^\infty [2k, 2k+1]$ is exponentially asymptotically stable, meanwhile $\limsup_{t \to \infty} \widehat{\Re}(-2)(t) = 0$. This indicates that, in general, the inverse of Theorem 2.28 is not true.

Conclusions of Chapter 2. By studying the ratio $\dfrac{|f(t)|}{e_\alpha(t,t_0)}$ with parameter α, as $t \to \infty$ we have overcame the difficulty when cannot define the logarithm function on time scales, and obtained some following results:

1. Introducing the Lyapunov exponent $\kappa_L[f(\cdot)]$ of the function $f : \mathbb{T}_{t_0} \to \mathbb{K}$, and obtaining the sufficient and necessary condition for the existence of $\kappa_L[f(\cdot)]$, Lemma 2.2, and as well as its basic properties;

2. Establishing the sufficient condition on the boundedness of Lyapunov exponent $\kappa_L[x(\cdot)]$, where $x(\cdot)$ is a nontrivial solution of dynamic equation $x^\Delta = A(t)x$ in Theorem 2.15. Besides that, we also obtain the Lyapunov's Inequality in Theorem 2.19;

3. Recommending the necessary and sufficient conditions for the exponential stability of equation $x^\Delta = A(t)x$ in Theorem 2.24 when $A(\cdot)$ is bounded, and deriving the spectral charaterization for the exponential stability in Theorem 2.25, as well as the sufficient conditions for the asymptotic stability in Theorems 2.26 and 2.28, where $A(\cdot)$ is constant matrix.

The results obtained in Chapter 2 are only preliminary studies of Lyapunov exponent for the homogeneous linear systems. We hope to achieve sharper results for linear dynamic systems, and especially, the ones for linearized systems on time scales.

CHAPTER 3

BOHL EXPONENTS FOR IMPLICIT DYNAMIC EQUATIONS

Consider linear time-varying implicit dynamic equations (IDEs) of the form

$$E_\sigma(t)x^\Delta(t) = A(t)x(t),\ t \geq 0,$$

where $E_\sigma(\cdot)$ $A(\cdot)$ are continuous matrix funtions, $E_\sigma(\cdot)$ is supposed to be singular. If this equation is subject to an external force $f(t)$, then it becomes

$$E_\sigma(t)x^\Delta(t) = A(t)x(t) + f(t),\ t \geq 0.$$

In this chapter, we will define the notion of Bohl exponent for linear time-varying IDE with index-1 and investigate the relation between the exponential stability and Bohl exponent as well as the robustness of Bohl exponent when this equation is subject to perturbations acting on only the right-hand side or on both sides. The content of Chapter 3 is based on the paper No.2 and No.3 in the list of the author's scientific works.

3.1 Linear Implicit Dynamic Equations with index-1

Consider linear time-varying implicit dynamic equation on time scales

$$E_\sigma(t)x^\Delta(t) = A(t)x(t) + f(t),\ \text{for all } t \geq \mathbb{T}_a, \tag{3.1}$$

where $A(\cdot)$, $E_\sigma(\cdot)$ are in $L_\infty^{\text{loc}}(\mathbb{T}_a, \mathbb{K}^{n\times n})$. Assume that $\operatorname{rank} E(t) = r$, $1 \leq r < n$, for all $t \in \mathbb{T}_a$ and $\ker E(t)$ is smooth in the sense that there exists a projector $Q(t)$ onto $\ker E(t)$ such that $Q(t)$ is continuously differentiable for all $t \in (a, \infty)$, $Q^2(t) = Q(t)$ and $Q^\Delta \in L_\infty^{\text{loc}}(\mathbb{T}_a, \mathbb{K}^{n\times n})$. Set $P(t) = I - Q(t)$.

It is clear that $P(t)$ is a projector along $\ker E(t)$, $P^2(t) = P(t)$ and we have $EP = E$. Then, Equation (3.1) can be rewritten in the form

$$E_\sigma(t)(Px)^\Delta(t) = \bar{A}(t)x(t) + f(t),\ t \geq a, \tag{3.2}$$

where $\bar{A} := A + E_\sigma P^\Delta \in L_\infty^{\mathrm{loc}}(\mathbb{T}_a; \mathbb{K}^{n\times n})$.

Let H be a continuous function defined on $\mathbb{T}_a$, taking values in the group $\mathrm{Gl}(\mathbb{R}^n)$ such that $H|_{\ker E_\sigma}$ is an isomorphism between $\ker E_\sigma$ and $\ker E$. We define the matrix $G := E_\sigma - \bar{A}HQ_\sigma$, and the set $S := \{x : Ax \in \operatorname{im} E_\sigma\}$.

Lemma 3.1. *The following assertions are equivalent.*

i) $S \cap \ker E = \{0\}$;

ii) *G is a nonsingular matrix;*

iii) $\mathbb{R}^n = S \oplus \ker E$.

Proof. See [22, Lemma 2.1]. □

Following from Lemma 3.1.ii), suppose that matrix G is nonsingular, we have the following lemmas.

Lemma 3.2. *There hold the following relations.*

i) $P_\sigma = G^{-1}E_\sigma$;

ii) $G^{-1}\bar{A}HQ_\sigma = -Q_\sigma$;

iii) *$\widetilde{Q} := -HQ_\sigma G^{-1}\bar{A}$ is the projector along S onto $\ker E$. We call $\widetilde{Q}$ the canonical projector, and $\widetilde{P} := I - \widetilde{Q}$;*

iv) *Let $\widehat{Q}$ be an arbitrary projector onto $\ker E$, and $\widehat{P} := I - \widehat{Q}$. Then, we have*

$$P_\sigma G^{-1}\bar{A} = P_\sigma G^{-1}\bar{A}\widehat{P},\ Q_\sigma G^{-1}\bar{A} = Q_\sigma G^{-1}\bar{A}\widehat{P} - H^{-1}\widehat{Q}.$$

Proof. See [22, Lemma 2.2]. □

Lemma 3.3. *The matricies $P_\sigma G^{-1}$, $HQ_\sigma G^{-1}$ do not depend on the choice of operators H, and Q.*

Proof. Let Q, Q' be two arbitrary projectors onto $\ker E(t)$, and set $P = I - Q$, $P' = I - Q'$, respectively. Let H, H' be two operators in $\mathrm{Gl}(\mathbb{R}^n)$ such that $H|_{\ker E_\sigma}, H'|_{\ker E_\sigma}$ are the isomorphisms between $\ker E_\sigma$ and $\ker E$, and $G' := E_\sigma - \bar{A}H'Q'_\sigma$. Then, we have

$$G^{-1}G' = G^{-1}(E_\sigma - \bar{A}H'Q'_\sigma) = P_\sigma - G^{-1}\bar{A}HH^{-1}H'Q'_\sigma.$$

Note that $\mathrm{im}(H'Q'_\sigma) = \ker E$ and $\mathrm{im}(H^{-1}H'Q'_\sigma) = \ker E_\sigma$, so $H^{-1}H'Q'_\sigma = Q_\sigma H^{-1}H'Q'_\sigma$, and then $P_\sigma H^{-1}H'Q'_\sigma = 0$. Hence,

$$G^{-1}G' = P_\sigma - G^{-1}\bar{A}HQ_\sigma H^{-1}H'Q'_\sigma = P_\sigma + Q_\sigma H^{-1}H'Q'_\sigma = P_\sigma + H^{-1}H'Q'_\sigma,$$

and we obtain

$$G^{-1} = (P_\sigma + H^{-1}H'Q'_\sigma)G'^{-1}.$$

Therefore,

$$P_\sigma G^{-1} = P_\sigma(P_\sigma + H^{-1}H'Q'_\sigma)G'^{-1} = P_\sigma G'^{-1},$$

and

$$HQ_\sigma G^{-1} = HQ_\sigma(P_\sigma + H^{-1}H'Q'_\sigma)G'^{-1} = HQ_\sigma H^{-1}H'Q'_\sigma G'^{-1} = H'Q'_\sigma G'^{-1}.$$

The proof is complete. □

Definition 3.4. The IDE (3.1) is said to be index-1 tractable on $\mathbb{T}_a$ if $G(t)$ is invertible for almost $t \in \mathbb{T}_a$ and $G^{-1} \in L^{\mathrm{loc}}_\infty(\mathbb{T}_a; \mathbb{K}^{n\times n})$.

Remark 3.5. According to Lemma 3.1, the index-1 property is independent of the choice of projector Q and the isomorphism H, see also [31, 57].

Let $J \subset \mathbb{T}$ be an interval. We denote the set

$$\mathrm{C}^1(J, \mathbb{K}^n) := \left\{ \begin{array}{l} x(\cdot) \in \mathrm{C}_{\mathrm{rd}}(J, \mathbb{K}^n) : P(t)x(t) \text{ is} \\ \text{delta-differentiable, almost } t \in J \end{array} \right\}.$$

Note that, $\mathrm{C}^1(J, \mathbb{K}^n)$ does not depend on the choice of projector functions. Since $P(t), \widehat{P}(t)$ are projectors along $\ker E$, we have

$$P(t)\widehat{P}(t) = P(t) \text{ and } \widehat{P}(t)P(t) = \widehat{P}(t).$$

Definition 3.6. The function $x(\cdot)$ is said to be a solution of Equation (3.1) (having index-1) on the interval J if $x(\cdot) \in \mathrm{C}^1(J, \mathbb{K}^n)$ and satisfies Equation (3.1) for almost $t \in J$.

Note that, we look for a solution $x(\cdot)$ of Equation (3.1) in the function space $C^1(J, \mathbb{K}^n)$. So $x(\cdot)$ is not neccessarily delta-differentiable. Since

$$E_\sigma x^\Delta = E_\sigma P_\sigma x^\Delta = E_\sigma(P^\Delta x + P_\sigma x^\Delta - P^\Delta x) = E_\sigma((Px)^\Delta - P^\Delta x),$$

we agree to use the expression $E_\sigma x^\Delta$ which stands for $E_\sigma((Px)^\Delta - P^\Delta x)$.

Multiplying both sides of Equation (3.2) by $P_\sigma G^{-1}$ and $Q_\sigma G^{-1}$, respectively, we decouple the index-1 Equation (3.2) into the following system

$$\begin{cases} (Px)^\Delta = (P^\Delta + P_\sigma G^{-1}\bar{A})Px + P_\sigma G^{-1} f, \\ Qx \quad\; = HQ_\sigma G^{-1}\bar{A}Px + HQ_\sigma G^{-1} f. \end{cases}$$

Since $x = (P+Q)x = Px + Qx$, we use the variable changes $u := Px$ and $v := Qx$ and get

$$u^\Delta = (P^\Delta + P_\sigma G^{-1}\bar{A})u + P_\sigma G^{-1} f, \tag{3.3}$$

$$v = HQ_\sigma G^{-1}\bar{A}u + HQ_\sigma G^{-1} f. \tag{3.4}$$

It means that Equation (3.2) is decomposed into two sub-equations, the delta-differential part (3.3) and the algebraic one (3.4). It is clear that we can solve u from Equation (3.3), then using Equation (3.4) to compute v. Finally, $x = u + v$. Therefore, we only need to address the initial value condition to the differential component (3.3). Let $t_0 \geq a$. Inspired by the above decoupling procedure, we state the initial condition $u(t_0) = P(t_0)x(t_0)$, or equivalent to

$$P(t_0)(x(t_0) - x_0) = 0, \; x_0 \in \mathbb{K}^n. \tag{3.5}$$

Remark 3.7. Multiplying both sides of Equation (3.3) by Q_σ yields $Q_\sigma u^\Delta = Q_\sigma P^\Delta u$. Noting that $(Qu)^\Delta = Q^\Delta u + Q_\sigma u^\Delta$ and $0 = (QP)^\Delta = Q^\Delta P + Q_\sigma P^\Delta$ comes to $(Qu)^\Delta = Q^\Delta Qu$. Thus, if $Q(t_0)u(t_0) = 0$ then $Q(t)u(t) = 0$, for all $t \in \mathbb{T}_{t_0}$. This means that at the time point t, every solution starting in $\operatorname{im} P(t_0)$ remains in $\operatorname{im} P(t)$.

Consider the homogeneous case, i.e., $f(t) = 0$,

$$E_\sigma(t)x^\Delta(t) = A(t)x(t), \tag{3.6}$$

with initial condition $P(t_0)(x(t_0) - x_0) = 0$. The Cauchy operator $\Phi(t,s)$ generated by Equation (3.6) is defined by

$$\begin{cases} E_\sigma(t)\Phi^\Delta(t,s) = A(t)\Phi(t,s), \\ P(s)(\Phi(s,s) - I) = 0. \end{cases}$$

We can solve the Cauchy operator $\Phi(t,s)$ by using the canonical projector $\widetilde{Q}(t) = -H(t)Q_\sigma(t)G^{-1}(t)\bar{A}(t)$ in Lemma 3.2. Let $\widetilde{P}(t) = I - \widetilde{Q}(t) = I + H(t)Q_\sigma(t)G^{-1}(t)\bar{A}(t)$, and $\Phi_0(t,s)$ denote the Cauchy operator generated by the system

$$\begin{cases} \Phi_0^\Delta(t,s) = (P^\Delta(t) + P_\sigma(t)G^{-1}(t)\bar{A}(t))\Phi_0(t,s), \\ \Phi_0(s,s) = I. \end{cases}$$

Then, the Cauchy operator of Equation (3.6) is defined as follows

$$\Phi(t,s) = \widetilde{P}(t)\Phi_0(t,s)P(s). \tag{3.7}$$

By Lemma 3.2 and Remark 3.7, we see that

$$P(t)\Phi(t,s) = P(t)\widetilde{P}(t)\Phi_0(t,s)P(s) = \Phi_0(t,s)P(s), \tag{3.8}$$

and hence,

$$\Phi(r,t)\Phi(t,s) = \Phi(r,s).$$

By using variation of constants formula, the unique solution of Equation (3.3) is defined by

$$u(t) = \Phi_0(t,t_0)u(t_0) + \int_{t_0}^{t} \Phi_0(t,\sigma(s))P_\sigma(s)G^{-1}(s)f(s)\Delta s. \tag{3.9}$$

Moreover, by (3.4), (3.7), (3.8), and (3.9) we have

$$\begin{aligned} u(t) + v(t) &= (I + H(t)Q_\sigma(t)G^{-1}(t)\bar{A}(t))\Phi_0(t,t_0)u(t_0) \\ &\quad + (I + H(t)Q_\sigma(t)G^{-1}(t)\bar{A}(t)) \int_{t_0}^{t} \Phi_0(t,\sigma(s))P_\sigma(s)G^{-1}(s)f(s)\Delta s \\ &\quad + H(t)Q_\sigma(t)G^{-1}(t)f(t) \\ &= \widetilde{P}(t)\Phi_0(t,t_0)P(t_0)x_0 + \int_{t_0}^{t} \widetilde{P}(t)\Phi_0(t,\sigma(s))P_\sigma(s)G^{-1}(s)f(s)\Delta s \\ &\quad + H(t)Q_\sigma(t)G^{-1}(t)f(t). \end{aligned}$$

Therefore, the unique solution of the initial value problem for the IDE (3.1) is

$$\begin{aligned} x(t) = \Phi(t,t_0)P(t_0)x_0 + \int_{t_0}^{t} \Phi(t,\sigma(s))P_\sigma(s)G^{-1}(s)f(s)\Delta s \\ + H(t)Q_\sigma(t)G^{-1}(t)f(t). \end{aligned} \tag{3.10}$$

From now on, we suppose that the following assumption holds.

Assumption 3.1. *There exists a bounded differentiable projector Q onto* ker E. *Let us denote $P := I - Q$ and $K_0 := \sup_{t \geq a} \|P(t)\|$.*

3.2 Stability of IDEs under non-Linear Perturbations

Let $a \in \mathbb{T}$ be a fixed point. We consider the perturbation of the form $f(t) := F(t, x(t))$, where F is a function defined on $\mathbb{T}_a \times \mathbb{R}^n$ such that $F(t, 0) = 0$ for all $t \in \mathbb{T}_a$. Then Equation (3.1) is rewritten as follows

$$E_\sigma(t)x^\Delta(t) = A(t)x(t) + F(t, x(t)),\ t \geq a. \tag{3.11}$$

Since $F(t, 0) = 0$ for all $t \in \mathbb{T}_a$, Equation (3.11) has the trivial solution $x(t) \equiv 0$. By using variable changes $u := Px$, $v := Qx$, and transformation techniques in Section 3.1, we get the equations

$$u^\Delta = (P^\Delta + P_\sigma G^{-1}\bar{A})u + P_\sigma G^{-1}F(t, u + v), \tag{3.12}$$

$$v = HQ_\sigma G^{-1}\bar{A}u + HQ_\sigma G^{-1}F(t, u + v). \tag{3.13}$$

Assume that $HQ_\sigma G^{-1}F(t, \cdot)$ is Lipschitz continuous with Lipschitz constant $\gamma_t < 1$, i.e.,

$$\|HQ_\sigma G^{-1}F(t, y) - HQ_\sigma G^{-1}F(t, z)\| \leq \gamma_t\|y - z\|,$$

for all $t \geq a$. Since $HQ_\sigma G^{-1}$ does not depend on the choice of H and Q, the Lipschitz property of $HQ_\sigma G^{-1}F(t, \cdot)$ does, too.

Fix $u \in \mathbb{R}^n$ and choose $t \in \mathbb{T}_a$, we consider the mapping $\Gamma_t : \operatorname{im} Q(t) \to \operatorname{im} Q(t)$ defined by

$$\Gamma_t(v) := H(t)Q_\sigma(t)G^{-1}(t)\bar{A}(t)u + H(t)Q_\sigma(t)G^{-1}(t)F(t, u + v).$$

It can be seen directly that

$$\|\Gamma_t(v) - \Gamma_t(v')\| \leq \gamma_t\|v - v'\|$$

for all $v, v' \in \operatorname{im} Q(t)$. Since $\gamma_t < 1$, Γ_t is a contractive mapping. Therefore, by the Fixed-point Theorem, there exists a mapping $g_t : \operatorname{im} P(t) \to \operatorname{im} Q(t)$, given by

$$g_t(u) := H(t)Q_\sigma(t)G^{-1}(t)\big(\bar{A}(t)u + F(t, u + g_t(u))\big). \tag{3.14}$$

Moreover, we have

$$\|g_t(u) - g_t(u')\| \leq \beta_t\|u - u'\| + \gamma_t\big(\|u - u'\| + \|g_t(u) - g_t(u')\|\big),$$

where $\beta_t = \|H(t)Q_\sigma(t)G^{-1}(t)\bar{A}(t)\|$. Hence, we get

$$\|g_t(u) - g_t(u')\| \leq \frac{\gamma_t + \beta_t}{1 - \gamma_t}\|u - u'\|.$$

This proves that the function g_t is Lipschitz continuous with Lipschitz constant $L_t := \frac{\gamma_t+\beta_t}{1-\gamma_t}$. Substituting $v = g_t(u)$ into (3.12) obtains

$$u^\Delta = (P^\Delta + P_\sigma G^{-1}\bar{A})u + P_\sigma G^{-1}F\big(t, u + g_t(u)\big). \tag{3.15}$$

Suppose that (3.15) is solvable. We get the solution $u(t)$ from Equation (3.15). Therefore, the unique solution of Equation (3.11) is

$$x(t) = u(t) + g_t(u(t)),\ t \in \mathbb{T}_a. \tag{3.16}$$

Definition 3.8. The IDE (3.11) is said to be exponentially stable if there exist numbers $M > 0, \alpha > 0$ such that $-\alpha \in \mathcal{R}^+$ and

$$\|x(t, t_0, x_0)\| \leq Me_{-\alpha}(t, t_0)\|P(t_0)x_0\|, \text{ for all } t \geq t_0 \geq a,\ x_0 \in \mathbb{R}^n.$$

By the classical way, we see that the uniform boundedness and the exponential stability of Equation (3.6) are characterized by Cauchy operator $\Phi(t,s)$ as follows:

Theorem 3.9. *The implicit dynamic equation* (3.6) *is exponentially stable if and only if there exist numbers $M > 0$ and positively regressive $-\alpha$ such that*

$$\|\Phi(t,s)\| \leq Me_{-\alpha}(t,s),\ \textit{for all } t \geq s \geq a. \tag{3.17}$$

Proof. Similarly to the proof of [23, Theorem 3.14]. □

From the equality (3.8), and Assumption 3.1 we get

$$\|\Phi_0(t,s)P(s)\| = \|P(t)\Phi(t,s)\| \leq \|P(t)\|\|\Phi(t,s)\| \leq K_0\|\Phi(t,s)\|.$$

Thus, from the inequality (3.17), there exists a positive constant M such that

$$\|\Phi_0(t,s)P(s)\| \leq Me_{-\alpha}(t,s),\ \text{for all } t \geq s \geq a. \tag{3.18}$$

We are now in position to consider the robust stability of IDEs under small perturbations. The following theorem will show that the exponential stability is also preserved under some integrable perturbations or small enough Lipschitz perturbation.

Theorem 3.10. *Assume that Equation* (3.6) *is of* index-1, *exponential stable and*

i) $L = \sup_{t\in\mathbb{T}_a} L_t < \infty$, *and*

ii) *the function* $P_\sigma(t)G^{-1}(t)F(t,x)$ *is Lipschitz continuous with Lipschitz constant* k_t, *such that one of the following conditions hold*

a) $N = \displaystyle\int_a^\infty \frac{k_t}{1-\alpha\mu(t)}\Delta t < \infty.$

b) $\limsup_{t\to\infty} k_t(1+L_t) = \delta < \frac{\alpha}{LM}$, *with* α, M *in Definition* 3.8.

Then, there exist the constants $K > 0$ *and positively regressive* $-\alpha_1$ *such that*

$$\|x(t)\| \le Ke_{-\alpha_1}(t,s)\|P(s)x(s)\|,$$

for all $t \ge s \ge a$, *where* $x(\cdot)$ *is a solution of* (3.11). *That is, the perturbed equation* (3.11) *preserves the exponential stability.*

Proof. We now prove this theorem with the condition a). By the variation of constants formula (3.9), the solution of Equation (3.15) is

$$u(t) = \Phi_0(t,s)u(s) + \int_s^t \Phi_0(t,\sigma(\tau))P_\sigma G^{-1}F(\tau, u(\tau) + g_\tau(u(\tau)))\Delta\tau,$$

for all $t > s \ge a$. Therefore, by estimate (3.18) we have

$$\begin{aligned} \|u(t)\| &\le \|\Phi_0(t,s)u(s)\| \\ &\quad + \int_s^t \|\Phi_0(t,\sigma(\tau))P_\sigma\| \|P_\sigma G^{-1}F(\tau, u(\tau) + g_\tau(u(\tau)))\|\Delta\tau \\ &\le Me_{-\alpha}(t,s)\|u(s)\| + M\int_s^t e_{-\alpha}(t,\sigma(\tau))k_\tau(1+L_\tau)\|u(\tau)\|\Delta\tau. \end{aligned} \tag{3.19}$$

Multiplying both sides of (3.19) by $\frac{1}{e_{-\alpha}(t,s)}$ yields

$$\frac{\|u(t)\|}{e_{-\alpha}(t,s)} \le M\|u(s)\| + M\int_s^t \frac{k_\tau(1+L_\tau)\|u(\tau)\|}{(1-\alpha\mu(\tau))e_{-\alpha}(\tau,s)}\Delta\tau. \tag{3.20}$$

Using Gronwall's inequality, we obtain

$$\frac{\|u(t)\|}{e_{-\alpha}(t,s)} \le M\|u(s)\|e_{\frac{M(1+L)k_\cdot}{1-\alpha\mu(\cdot)}}(t,s),$$

Since $\frac{M(1+L)k_{\cdot}}{1-\alpha\mu(\cdot)}$ is positive, by the definition of the exponential funtion it follows that

$$e_{\frac{M(1+L)k_{\cdot}}{1-\alpha\mu(\cdot)}} \leq e^{\int_s^t \frac{M(1+L)k_\tau}{1-\alpha\mu(\tau)}\Delta\tau} \leq e^{\int_s^\infty \frac{M(1+L)k_\tau}{1-\alpha\mu(\tau)}\Delta\tau} = e^{MN(1+L)}.$$

Therefore, there exists a number $M_1 > 0$ such that

$$\|u(t)\| \leq M_1 e_{-\alpha}(t,s)\|u(s)\|.$$

By (3.16) we get

$$\|x(t)\| \leq \|u(t)\| + \|g_t(u(t))\| \leq (1+L)\|u(t)\| \leq (1+L)M_1 e_{-\alpha}(t,s)\|u(s)\|,$$

or,

$$\|x(t)\| \leq K_0 e_{-\alpha}(t,s)\|P(s)x(s)\|,$$

for all $t > s \geq a$, where $K_0 = (1+L)M_1$. We have the proof in the first case.

Next, in the case condition b) is satisfied, let ε_0 be a positive number such that $\delta + \varepsilon_0 \leq \frac{\alpha}{LM}$. Then, follow from the second assumption, there exists an element $T_0 > a$ such that

$$k_t(1+L_t) < \delta + \varepsilon_0, \text{ for all } t > T_0. \tag{3.21}$$

By the solutions' continuity of (3.15) with the initial condition, we can find a positive constant M_{T_0} which depends only on T_0 such that

$$\|u(t)\| \leq M_{T_0}\|u(s)\|, \text{ for all } a \leq s < t \leq T_0. \tag{3.22}$$

First, we consider the case $t > T_0 > s \geq a$. In the same way as (3.19) and (3.20), it follows that

$$\frac{\|u(t)\|}{e_{-\alpha}(t,s)} \leq M\|u(T_0)\| e_{\frac{Mk_{\cdot}(1+L_{\cdot})}{1-\alpha\mu(\cdot)}}(t,T_0) \leq M\|u(T_0)\| e_{\frac{M(\delta+\varepsilon_0)}{1-\alpha\mu(\cdot)}}(t,T_0).$$

or equivalently,

$$\|u(t)\| \leq M\|u(T_0)\| e_{-\alpha \oplus \frac{M(\delta+\varepsilon_0)}{1-\alpha\mu(\cdot)}}(t,T_0) = M\|u(T_0)\| e_{-\alpha + M(\delta+\varepsilon_0)}(t,T_0).$$

It is clear that $L > 1$. Set $\alpha_1 := \alpha - M(\delta+\varepsilon_0) > 0$. Since $-\alpha$ is positively regressive, so is $-\alpha_1$. Therefore, by (3.22), we have

$$\|u(t)\| \leq \frac{M\|u(T_0)\|}{e_{-\alpha_1}(T_0,s)} e_{-\alpha_1}(t,s) \leq M e_{\ominus(-\alpha_1)}(T_0,t_0) e_{-\alpha_1}(t,s)\|u(T_0)\|,$$

or

$$\|u(t)\| \le MM_{T_0}e_{\ominus(-\alpha_1)}(T_0,t_0)e_{-\alpha_1}(t,s)\|u(s)\|.$$

Thus,

$$\|u(t)\| \le K_1 e_{-\alpha_1}(t,s)\|u(s)\|,$$

where $K_1 = MM_{T_0}e_{\ominus(-\alpha_1)}(T_0,t_0)$.

Next, we consider the case $t > s \ge T_0$. We have the estimate

$$\|P_\sigma G^{-1}F(t,u+g_t(u))\| \le (\delta+\varepsilon_0)\|u\|$$

for all $t \ge s$. Therefore, by the similar arguments as above, we obtain

$$\|u(t)\| \le K_2\|u(s)\|e_{-\alpha_1}(t,s).$$

Finally, we consider the remaining case $a \le s < t \le T_0$. With the number $\alpha_1 > 0$ defined above, we have

$$\|u(t)\| \le M_{T_0}\|u(s)\| \le M_{T_0}e_{\alpha_1}(T_0,t_0)e_{-\alpha_1}(t,s)\|u(s)\|.$$

Put $K_3 = \max\{K_1, K_2, M_{T_0}e_{\alpha_1}(T_0,t_0)\}$, we get

$$\|u(t)\| \le K_3 e_{-\alpha_1}(t,s)\|u(s)\|.$$

Pay attention to (3.16), we obtain

$$\|x(t)\| \le Ke_{-\alpha_1}(t,s)\|P(s)x(s)\|,$$

for all $t \ge s \ge a$, where $K = (1+L)K_3$. The proof is complete. □

Remark 3.11. If E_σ is the identity matrix then from Theorem 3.10 we can obtain results about robust stability of the dynamic systems on time scales $x^\Delta(t) = A(t)x(t) + F(t,x)$ in [29].

Remark 3.12. Assume that the perturbation $F(t,x)$ is linear, i.e. $F(t,x) = \Sigma(t)x$ with $\Sigma(t) \in \mathbb{R}^{n\times n}$. Then the perturbed equation (3.11) has the form

$$E_\sigma x^\Delta = (A(t) + \Sigma(t))x(t).$$

In this case, it is easy to see that $\gamma(t) = \|H(t)Q_\sigma(t)G^{-1}(t)\Sigma(t)\| < 1$ if $\Sigma(t)$ is small enough and $k_t = \|P_\sigma(t)G^{-1}(t)\Sigma(t)\|$. By Theorem 3.10, we can derive bounds for the perturbation $\Sigma(t)$ such that the perturbed equation (3.11) is still exponentially stable. This can be used to evaluate the robust stability of DAEs, respectively $\mathbb{T} = \mathbb{R}$, and implicit difference equations, respectively $\mathbb{T} = \mathbb{Z}$, which arise in many applications, see [46, 48, 51, 55].

Next, we prove Bohl-Perron type theorem for linear time-varying IDEs, i.e., investigate the relation between the solutions' boundedness of the nonhomogenous Equation (3.1) and the exponential stability of the IDE (3.6).

We note that, in solving equation (3.1), the function f is split into two components $P_\sigma G^{-1} f$ and $HQ_\sigma G^{-1} f$. Therefore, for any $t_0 \in \mathbb{T}_a$ we consider the function f as an element of the set

$$L(t_0) = \left\{ \begin{array}{r} f \in C([t_0, \infty], \mathbb{R}^n) : \sup_{t \geq t_0} \|H(t)Q_\sigma(t)G^{-1}(t)f(t)\| < \infty \\ \text{and } \sup_{t \geq t_0} \|P_\sigma(t)G^{-1}(t)f(t)\| < \infty \end{array} \right\}.$$

We can directly see that $L(t_0)$ is a Banach space eqiupped with the norm

$$\|f\| = \sup_{t \geq t_0} \left(\|P_\sigma(t)G^{-1}(t)f(t)\| + \|H(t)Q_\sigma(t)G^{-1}(t)f(t)\| \right).$$

Denote by $x(t, s, f)$ the solution, associated with f, of Equation (3.1) with the initial condition $P(s)x(s, s) = 0$. For notational convenience, we will write $x(t, s)$ or $x(t)$ for $x(t, s, f)$ if that causes no confusion, and IVP-1 for "Equation (3.1) with the initial condition $P(s)x(s, s) = 0$".

Lemma 3.13. *If for any function $f(\cdot) \in L(t_0)$, the solution $x(\cdot, t_0)$ of the IVP-1 is bounded, then for all $t_1 \geq t_0$, there is a constant $k > 0$, independent of t_1, such that* $\sup_{t \geq t_1} \|x(t, t_1)\| \leq k\|f\|$.

Proof. Define a family of operators $\{V_t\}_{t \geq t_0}$ as follows

$$\begin{aligned} V_t : L(t_0) &\longrightarrow \mathbb{R}^n \\ f &\longmapsto V_t(f) = x(t, t_0). \end{aligned}$$

From the assumption of this lemma, we have $\sup_{t \geq t_0} \|V_t f\| < \infty$ for any $f \in L(t_0)$. By the Uniform Boundedness Principle, there exists a constant $k > 0$ such that

$$\sup_{t \geq t_0} \|x(t, t_0)\| = \|V_t f\| \leq k\|f\|, \tag{3.23}$$

for all $t \geq t_0$. Let f be an arbitrary funtion in $L(t_1)$. We define a function $\bar{f}$ in $L(t_0)$ as follows: if $t < t_1$ then $\bar{f}(t) = 0$, else $\bar{f}(t) = f(t)$. Hence, by the variation of constants formula, for any $t \geq t_1$ we have

$$x(t, t_0, \bar{f}) = \int_{t_0}^{t} \Phi(t, \sigma(\tau))P_\sigma(\tau)G^{-1}(\tau)\bar{f}(\tau)d\tau + H(t)Q_\sigma(t)G^{-1}(t)\bar{f}(t)$$

$$= \int_{t_1}^{t} \Phi(t, \sigma(\tau)) P_\sigma(\tau) G^{-1}(\tau) f(\tau) d\tau + H(t) Q_\sigma(t) G^{-1}(t) f(t).$$

This means that

$$x(t, t_0, \bar{f}) = x(t, t_1, f) \text{ for all } t \geq t_1 \geq t_0.$$

Therefore, from (3.23) we have the relation

$$\sup_{t \geq t_1} \|x(t, t_1, f)\| = \sup_{t \geq t_0} \|x(t, t_0, \bar{f})\| \leq k\|\bar{f}\| = k\|f\|.$$

The proof is complete. □

We are now in the position to derive the Bohl-Perron type stability theorem for linear time-varying IDEs.

Theorem 3.14. *All solutions of the IVP-1, associated with an arbitrary function f in $L(t_0)$, are bounded if and only if the index-1 IDE* (3.6) *is exponentially stable.*

Proof. The proof is divided into two parts.

Necessity. We prove that if all solutions of the IVP-1 associated with $f \in L(t_0)$, are bounded, then the IDE (3.6) is exponentially stable. Indeed, for any $t_1 \geq t_0$, let

$$\chi(t) := \|\Phi(\sigma(t), t_1)\|, \ t \geq t_1.$$

For any $y \in \mathbb{R}^n$, we consider the function

$$f(t) = \frac{E_\sigma(t)\Phi(\sigma(t), t_1)y}{\chi(t)}, \ t \geq t_1.$$

It is obvious that

$$\begin{aligned} \|P_\sigma(t) G^{-1}(t) f(t)\| &= \left\| P_\sigma(t) G^{-1}(t) \frac{E_\sigma(t)\Phi(\sigma(t), t_1)y}{\chi(t)} \right\| \\ &= \left\| P_\sigma(t) \frac{\Phi(\sigma(t), t_1)}{\chi(t)} y \right\| \leq K_0 \|y\|. \end{aligned}$$

Moreover,

$$\|H(t) Q_\sigma(t) G^{-1}(t) f(t)\| = \left\| H(t) Q_\sigma(t) G^{-1}(t) \frac{E_\sigma(t)\Phi(\sigma(t), t_1)y}{\chi(t)} \right\| = 0.$$

Thus, $f \in L(t_1)$ and

$$\|f\| = \sup_{t \geq t_1} \left(\|P_\sigma(t) G^{-1}(t) f(t)\| + \|H(t) Q_\sigma(t) G^{-1}(t) f(t)\| \right) \leq K_0 \|y\|.$$

Moreover,

$$\begin{aligned} x(t, t_1) &= \int_{t_1}^{t} \Phi(t, \sigma(\tau)) P_\sigma(\tau) G^{-1}(\tau) f(\tau) \Delta\tau + H(t) Q_\sigma(t) G^{-1}(t) f(t) \\ &= \int_{t_1}^{t} \Phi(t, \sigma(\tau)) P_\sigma(\tau) \frac{\Phi(\sigma(\tau), t_1) y}{\chi(\tau)} \Delta\tau = \int_{t_1}^{t} \frac{\Phi(t, t_1) y}{\chi(\tau)} \Delta\tau. \end{aligned}$$

Put $\Psi(t) = \int_{t_1}^{t} \frac{1}{\chi(\tau)} \Delta\tau > 0$, we have $x(t, t_1) = \Phi(t, t_1)\Psi(t) y$. From Lemma 3.13, we obtain

$$\|x(t, t_1)\| = \|\Phi(t, t_1)\Psi(t) y\| = \|\Phi(t, t_1) y\| \Psi(t) \leq k\|f\| \leq kK_0\|y\|,$$

which implies that

$$\|\Phi(t, t_1)\| \leq \frac{h}{\Psi(t)}, \tag{3.24}$$

where $h = kK_0$. On the other hand,

$$\frac{1}{\Psi^\Delta(t)} = \chi(t) = \|\Phi(\sigma(t), t_1)\| \leq \frac{h}{\Psi(\sigma(t))}.$$

Therefore,

$$\Psi^\Delta(t) \geq \frac{1}{h} \Psi(\sigma(t)).$$

By Theorem 1.38, we get $\Psi(t) \geq \Psi(c) e_{\ominus(-\frac{1}{h})}(t, c)$, for all $t \geq c$. Hence, by (3.24) we have

$$\|\Phi(\sigma(t), t_1)\| \leq \frac{h}{\Psi(c)} e_{-\frac{1}{h}}(\sigma(t), c),$$

for all $t \geq c$. This estimate leads to

$$\|\Phi(t, t_1)\| \leq \frac{h}{\Psi(c)} e_{-\frac{1}{h}}(t, c) = \frac{h}{\Psi(c) e_{-\frac{1}{h}}(c, t_1)} e_{-\frac{1}{h}}(t, t_1)$$

for all $t \geq c$. Set $\alpha = \frac{1}{h}$, $N_1 = \frac{h}{\Psi(c) e_{-\frac{1}{h}}(c, t_1)}$ and

$$N = \max \left\{ N_1, \max_{t_1 \leq t \leq c} \frac{\|\Phi(t, t_1)\|}{e_{-\alpha}(t, t_1)} \right\},$$

we obtain the desired estimate $\|\Phi(t,t_1)\| \le Ne_{-\alpha}(t,t_1)$ for all $t \ge t_1$.

Sufficiency. To complete the proof, we will show that, if (3.6) is exponentially stable, then all solutions of the IVP-1 associated with f in $L(t_0)$, are bounded. Indeed, let $f \in L(t_0)$ and suppose that

$$\sup_{t\ge t_0} \|P_\sigma(t)G^{-1}(t)f(t)\| = C_1, \quad \sup_{t\ge t_0} \|H(t)Q_\sigma(t)G^{-1}(t)f(t)\| = C_2.$$

By the variant of constants formula (3.10), we have

$$\begin{aligned}\|x(t)\| &\le \int_{t_0}^{t} \|\Phi(t,\sigma(\tau))P_\sigma G^{-1}f(\tau)\|\Delta\tau + \|HQ_\sigma G^{-1}f(t)\| \\ &\le MC_1 \int_{t_0}^{t} e_{-\alpha}(t,\sigma(\tau))\Delta\tau + C_2 \\ &= MC_1 e_{-\alpha}(t,t_0) \int_{t_0}^{t} e_{\ominus(-\alpha)}(\sigma(\tau),t_0)\Delta\tau + C_2.\end{aligned}$$

By using the L'Hôspital rule, we get

$$\begin{aligned}\lim_{t\to\infty} e_{-\alpha}(t,t_0)\int_{t_0}^{t} e_{\ominus(-\alpha)}(\sigma(\tau),t_0)\Delta\tau &= \lim_{t\to\infty} \frac{\int_{t_0}^{t} e_{\ominus(-\alpha)}(\sigma(\tau),t_0)\Delta\tau}{e_{\ominus(-\alpha)}(t,t_0)} \\ &= \lim_{t\to\infty} \frac{e_{\ominus(-\alpha)}(\sigma(t),t_0)}{\ominus(-\alpha)e_{\ominus(-\alpha)}(t,t_0)} = \frac{1}{\alpha}.\end{aligned}$$

Thus, $\sup_{t\ge t_0} \int_{t_0}^{t} e_{-\alpha}(t,\sigma(\tau))\Delta\tau < \infty$, which implies that solutions of Equation (3.1) associated with f are bounded. The proof is complete. □

Remark 3.15. The above results extended the Bohl-Perron type stability theorem with bounded input/output operators for differential and difference equations [3, 18, 64], for differential-algebraic equations [34], and implicit difference equations [26, 52], corresponding to the case $\mathbb{T} = \mathbb{R}$ or $\mathbb{T} = \mathbb{Z}$ for dynamic systems on time scales [29].

Example 3.16. Consider the simple circuit on time scales consists of a voltage source $v_V = v(t)$, a resistor with conductance R and a capacitor with capacitance $C > 0$, see Figure 3.1. As in [71], this model can be written in the form $E_\sigma x^\Delta = Ax + f$, with

$$E_\sigma = \begin{bmatrix} 0 & 0 & 0 \\ 0 & C & 0 \\ 0 & 0 & 0 \end{bmatrix}, \; A = \begin{bmatrix} -R & R & 1 \\ R & -R & 0 \\ 1 & 0 & 0 \end{bmatrix}, \; x = \begin{bmatrix} e_1 \\ e_2 \\ i_v \end{bmatrix}, \; f = \begin{bmatrix} 0 \\ 0 \\ v \end{bmatrix}.$$

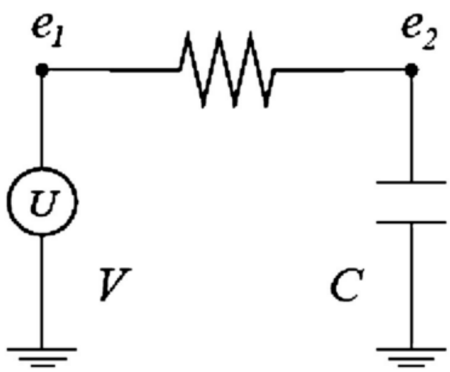

Figure 3.1: A simple circuit

In this case, it is easy to see that

$$Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \ P = I - Q = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \ H = I.$$

Therefore,

$$G = E_\sigma - \bar{A}HQ_\sigma = \begin{bmatrix} R & 0 & -1 \\ -R & C & 0 \\ -1 & 0 & 0 \end{bmatrix}, \ G^{-1} = \begin{bmatrix} 0 & 0 & -1 \\ 0 & \frac{1}{C} & \frac{-R}{C} \\ -1 & 0 & -R \end{bmatrix}.$$

This implies that $\|f\| = \sqrt{1 + \frac{R^2(C^2+1)}{C^2}}\|v\|$. On the other hand, the spectral set

$$\sigma(E_\sigma, A) = \{\lambda : \det(A - \lambda E_\sigma) = 0\} = \left\{\frac{-R}{C}\right\}.$$

Therefore, if $1 - \frac{\mu(t)R}{C} > 0$, or equivalently $\frac{-R}{C} \in \mathcal{R}^+$ then the homogenous equation $E_\sigma x^\Delta = Ax$ is exponentially stable. By Theorem 3.14, if v is bounded then e_1, e_2, i_v are bounded.

3.3 Bohl Exponent for Implicit Dynamic Equations

In this section, we extend the concept of Bohl exponent for the linear time-varying IDEs on time scale $\mathbb{T}$ and consider the robustness of Bohl exponent when these equations are subject to perturbations acting on the right side, or both sides. This will generalize and unify some results about the Bohl exponent for DAEs in [6, 14] and for implict difference equations in [26].

3.3.1 Bohl Exponent: Definition and Property

Definition 3.17. Assume that the IDE (3.6) is of index-1, and $\Phi(t,s)$ is its Cauchy operator. Then, the (upper) Bohl exponent of the IDE (3.6) is defined by

$$\kappa_{\mathcal{B}}(E,A) = \inf\{\alpha \in \mathbb{R}; \exists M_\alpha > 0 : \|\Phi(t,s)\| \le M_\alpha e_\alpha(t,s), \text{ for all } t \ge s \ge t_0\}.$$

When $\kappa_{\mathcal{B}}(E,A) = -\frac{1}{\mu^*}$ or $\kappa_{\mathcal{B}}(E,A) = +\infty$, the Bohl exponent of the IDE (3.6) is called *extreme*. Specially, in case $\mathbb{T} = \mathbb{R}$ (resp. $\mathbb{T} = h\mathbb{Z}$), we come to the classical definition of Bohl exponent, and extreme exponents may be $\pm\infty$ (resp. $-\frac{1}{h}$ or $+\infty$). Further,

Proposition 3.18. *If $\alpha = \kappa_{\mathcal{B}}(E,A)$ is not extreme, then for any $\varepsilon > 0$ we have*

i) $\displaystyle\lim_{\substack{t-s\to\infty\\ s\to\infty}} \frac{\|\Phi(t,s)\|}{e_{\alpha\oplus\varepsilon}(t,s)} = 0$ ii) $\displaystyle\limsup_{\substack{t-s\to\infty\\ s\to\infty}} \frac{\|\Phi(t,s)\|}{e_{\alpha\ominus\varepsilon}(t,s)} = \infty.$

Proof. i) By Definition 3.17, there exists a number $M_\alpha > 0$ such that

$$\|\Phi(t,s)\| \le M_\alpha e_\alpha(t,s).$$

Multiplying both sides of this inequality by $e_\varepsilon(t,s)$, we get

$$\|\Phi(t,s)\| e_\varepsilon(t,s) \le M_\alpha e_{\alpha\oplus\varepsilon}(t,s),$$

or

$$\frac{\|\Phi(t,s)\|}{e_{\alpha\oplus\varepsilon}(t,s)} \le \frac{M_\alpha}{e_\varepsilon(t,s)} = M_\alpha e_{\ominus\varepsilon}(t,s) \le M_\alpha e^{-K(t-s)},$$

where $K = \frac{\varepsilon}{1+\mu^*\varepsilon}$. Therefore,

$$\lim_{\substack{t-s\to\infty\\ s\to\infty}} \frac{\|\Phi(t,s)\|}{e_{\alpha\oplus\varepsilon}(t,s)} \le \lim_{\substack{t-s\to\infty\\ s\to\infty}} \frac{M_\alpha}{e_\varepsilon(t,s)} = 0,$$

which deduces that

$$\lim_{\substack{t-s\to\infty\\ s\to\infty}} \frac{\|\Phi(t,s)\|}{e_{\alpha\oplus\varepsilon}(t,s)} = 0.$$

ii) We choose $\delta > 0$ such that $1 + \alpha\mu(t) > \dfrac{\delta(1+\varepsilon\mu(t))}{\varepsilon}$ is equivalent to

$$\alpha - \delta \ge \frac{\alpha - \varepsilon}{1+\varepsilon\mu(t)} = \alpha \ominus \varepsilon, \quad \text{for all } t \in \mathbb{T}.$$

By definition, there exists a sequence $\{t_n, s_n\}$, $s_n \to \infty$, and $t_n - s_n \to \infty$ such that

$$\lim_{n\to\infty} \frac{\|\Phi(t_n, s_n)\|}{e_{\alpha-\delta}(t_n, s_n)} = \infty.$$

Further,

$$\frac{e_{\alpha-\delta}(t_n, s_n)}{e_{\alpha\ominus\varepsilon}(t_n, s_n)} \geq 1, \text{ for all } n \in \mathbb{N}.$$

Therefore,

$$\lim_{n\to\infty} \frac{\|\Phi(t_n, s_n)\|}{e_{\alpha\ominus\varepsilon}(t_n, s_n)} = \lim_{n\to\infty} \frac{\|\Phi(t_n, s_n)\|}{e_{\alpha-\delta}(t_n, s_n)} \frac{e_{\alpha-\delta}(t_n, s_n)}{e_{\alpha\ominus\varepsilon}(t_n, s_n)} \geq \lim_{n\to\infty} \frac{\|\Phi(t_n, s_n)\|}{e_{\alpha-\delta}(t_n, s_n)} = \infty.$$

The proof is complete. □

Remark 3.19. From Proposition 3.18, it is clear that

i) If $\mathbb{T} = \mathbb{R}$, then $\kappa_{\mathcal{B}}(E, A) = \limsup_{s, t-s\to\infty} \frac{\ln \|\Phi(t, s)\|}{t - s}$.

ii) If $\mathbb{T} = h\mathbb{Z}$, then $\kappa_{\mathcal{B}}(E, A) = \frac{1}{h}\left(\limsup_{s, t-s\to\infty} \|\Phi(t, s)\|^{\frac{h}{t-s}} - 1\right)$.

Example 3.20. Consider Equation (3.6) with

$$E(t) = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \; A(t) = \begin{bmatrix} p(t) & p(t) & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

on time scale $\mathbb{T} = \bigcup_{k=0}^{\infty} \{3k\} \bigcup_{k=0}^{\infty} [3k+1, 3k+2]$, where

$$p(t) = \begin{cases} -\frac{1}{4} & \text{if } t = 3k, \\ -\frac{1}{2} & \text{if } t \in [3k+1, 3k+2]. \end{cases} \tag{3.25}$$

In this case, we can choose and compute that

$$P = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 \end{bmatrix}, H = I, G^{-1} = \begin{bmatrix} \frac{1}{2} & -1 & 0 \\ \frac{1}{2} & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \widetilde{P} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Simple calculations yield that the transition matrices of the equation (E, A) are given by

$$\Phi_0(t,s) = \begin{bmatrix} \frac{e_p(t,s)+1}{2} & \frac{e_p(t,s)-1}{2} & 0 \\ \frac{e_p(t,s)-1}{2} & \frac{e_p(t,s)+1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \ \Phi(t,s) = \begin{bmatrix} 0 & 0 & 0 \\ e_p(t,s) & e_p(t,s) & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

By the definition of exponential function, it can directly see that for all $m, n \in \mathbb{N}, m > n$

$$e_p(3m, 3n) = \left(1 - \frac{1}{4}\right)^{m-n} \left(1 - \frac{1}{2}\right)^{m-n} e^{-\frac{m-n}{2}} = e^{\left(-\frac{1}{2}+\ln\frac{3}{8}\right)(m-n)}.$$

Let $0 < \alpha < 1$ be a solution of the equation

$$2\ln(1-\alpha) - \alpha = -\frac{1}{2} + \ln\frac{3}{8}.$$

Then,

$$e^{\left(-\frac{1}{2}+\ln\frac{3}{8}\right)(m-n)} = e_{-\alpha}(3m, 3n).$$

Thus, by Definition 3.17, we have $\kappa_{\mathcal{B}}(E, A) = -\alpha$.

Remark 3.21. Since $\widetilde{P}(t) = \Phi(t,t)$, we can directly see that if the Bohl exponent of Equation (3.6) is finite, then the canonical projector $\widetilde{P}$ must be bounded.

Assumption 3.2. *The terms $P_\sigma G^{-1}$ and $HQ_\sigma G^{-1}$ are bounded from above by the constants K_3 and K_4, respectively, on $\mathbb{T}_{t_0}$.*

Remark 3.22. According to Lemma 3.3, the boundedness of $P_\sigma G^{-1}$ and of $\widetilde{Q}$ does not depend on the choice of projectors H and Q.

The following result gives us a relationship among the exponential stability, the Bohl exponent of Equation (3.6) and solutions of the IVP-1.

Theorem 3.23. *The following assertions are equivalent.*

i) *The IDE* (3.6) *is exponentially stable;*

ii) *The Bohl exponent $\kappa_{\mathcal{B}}(E, A)$ is negative;*

iii) *The Bohl exponent* $\kappa_{\mathcal{B}}(E, A)$ *is finite and for any* $p > 0$*, there exists a positive constant* K_p *such that*

$$\int_s^\infty \|\Phi(t,s)\|^p \Delta t \leq K_p, \text{ for all } t \geq s \geq t_0;$$

iv) *All solutions of the IVP-1 associated with* f *in* $L(t_0)$ *are bounded.*

Proof. The equivalence between assertions i) and ii) (resp., i) and iv)) are due to Theorem 3.9 (resp., Theorem 3.14). We prove the rest cases.

ii) $\Longrightarrow$ iii): Let $\kappa_{\mathcal{B}}(E, A) = -2\alpha < 0$. Then there exists a constant $M_\alpha > 0$ such that

$$\|\Phi(t,s)\| \leq M_\alpha e_{-\alpha}(t,s), \text{ for all } t \geq s \geq t_0.$$

Let $\beta = \beta(p) := \min\{p, 1\}$ and $\gamma = \gamma(p) := \min\{1, 2^{p-1}\}$. From Bernoulli's inequality, we see that if $0 \leq \alpha\mu^* \leq \frac{1}{2}$ then

$$1 - \gamma\alpha\mu(t) \leq (1 - \alpha\mu(t))^p \leq 1 - \beta\alpha\mu(t), \text{ for all } t \in \mathbb{T}_a,$$

which implies that

$$e_{-\gamma\alpha}(t,s) \leq e^p_{-\alpha}(t,s) \leq e_{-\beta\alpha}(t,s). \tag{3.26}$$

Therefore,

$$\int_s^\infty \|\Phi(t,s)\|^p \Delta t \leq M_\alpha^p \int_s^\infty e^p_{-\alpha}(t,s)\Delta t \leq M_\alpha^p \int_s^\infty e_{-\alpha\beta}(t,s)\Delta t = \frac{M_\alpha^p}{\alpha\beta}.$$

Thus, we obtain iii).

iii) $\Longrightarrow$ ii): From the first inequality of (3.26) we see that

$$\int_s^{s+T} e^p_{-\alpha}(t,s)\Delta t \geq \int_s^{s+T} e_{-\alpha\gamma}(t,s)\Delta t = \frac{1 - e_{\alpha\gamma}(s+T, s)}{\alpha\gamma}.$$

Hence, we can choose $\alpha, T > 0$ such that

$$K_p < \inf_{s > t_0} \int_s^{s+T} e^p_{-\alpha}(t,s)\Delta t.$$

For $s = s_0$, since

$$\int_{s_0}^{s_0+T} \|\Phi(t,s)\|^p \Delta t \leq K_p < \int_{s_0}^{s_0+T} e_{-\alpha}(t,s)^p \Delta t,$$

we can define the element

$$s_1 = \max\{t : s_0 < t \le s_0 + T, \|\Phi(t, s_0)\| \le e_{-\alpha}(t, s_0)\}.$$

Similarly, we define a sequence $\{s_k\}$ as follows

$$s_{k+1} = \max\{t : s_k < t \le s_k + T, \|\Phi(t, s_k)\| \le e_{-\alpha}(t, s_k)\}.$$

It can be seen directly that $s_{k+1} > s_k$ and $s_{k+1} > s_{k-1} + T$ and $s_{k+1} - s_k \le T$, for all $k \in \mathbb{N}$. Therefore,

$$\inf_k e_{-\alpha}(s_{k+1}, s_k) := \alpha_1 > 0.$$

For $t \in [s_k, s_{k+1})$, we have

$$\begin{aligned}\|\Phi(t,s)\| =& \|\Phi(t,s_0)\| \le \|\Phi(t,s_k)\| \|\Phi(s_k,s_0)\| \\ &\le \left(\sup_{s_k \le t < s_k+T} \|\Phi(t,s_k)\|\right) \|\Phi(s_k,s_{k-1})\| \cdots \|\Phi(s_1,s_0)\|.\end{aligned}$$

Since $\kappa_{\mathcal{B}}(E, A) < \infty$, it is clear that there exists a constant $M_1 > 0$ such that $\|\Phi(t,s)\| \le M_1$, for all $t - s < T$. Therefore,

$$\|\Phi(t,s)\| \le M_1 e_{-\alpha}(s_k, s_0) \le \frac{M_1}{e_{-\alpha}(s_{k+1}, s_k)} e_{-\alpha}(t,s) \le \frac{M_1}{\alpha_1} e_{-\alpha}(t,s).$$

This implies that $\kappa_{\mathcal{B}}(E, A) < 0$. The proof is complete. □

Returning to Equation (3.6), we perfom the system transformation by changing variable $x(t) = U(t)z(t)$ and scaling both sides of Equation (3.6) by the matrix V, we come to a new equation

$$\widehat{E}_\sigma(t) z^\Delta(t) = \widehat{A}(t) z(t), \tag{3.27}$$

for all $t \ge t_0$, where $\widehat{E}_\sigma = V E_\sigma U_\sigma$, $\widehat{A} = V(AU - E_\sigma U^\Delta)$. Here, the matrices $U \in C^1(\mathbb{T}, \mathbb{R}^{n\times n})$, $V \in C(\mathbb{T}, \mathbb{R}^{n\times n})$ are pointwise non-singular matrix functions. Furthermore, we can directly verify that

$$\widehat{Q} = U^{-1}QU, \ \widehat{P} = U^{-1}PU, \ \widehat{H} = U^{-1}HU_\sigma, \ \widehat{G} = VGU_\sigma.$$

Hence, the Cauchy operator of (3.27) satisfies

$$\widehat{\Phi}(t,s) = U^{-1}(t)\Phi(t,s)U(s), \ \text{ for all } t \ge s \ge t_0. \tag{3.28}$$

Definition 3.24. With matrix functions $U \in C^1(\mathbb{T}, \mathbb{R}^{n\times n}), V \in C(\mathbb{T}, \mathbb{R}^{n\times n})$, the system transformation is said to be a Bohl transformation if

$$\inf\{\varepsilon \in \mathbb{R}; \exists M_\varepsilon > 0 : \|U^{-1}(t)\|\|U(s)\| \le M_\varepsilon e_\varepsilon(t,s), \text{ for all } t, s \ge t_0\} = 0.$$

The following results follow directly from Definition 3.24.

Proposition 3.25. i) *The set of Bohl transformations together with point multiplication forms a group.*

ii) *The Bohl exponent is invariant under Bohl transformations.*

3.3.2 Robustness of Bohl Exponents

Suppose that $\Sigma(\cdot) \in \mathbb{R}^{n\times n}$ is a continuous matrix function. We consider the perturbed equation

$$E_\sigma(t)x^\Delta(t) = (A(t) + \Sigma(t))x(t), \text{ for all } t \ge t_0. \tag{3.29}$$

It is clear that, Equation (3.29) is equivalent to

$$E_\sigma(t)(Px)^\Delta(t) = (\bar{A}(t) + \Sigma(t))x(t), \text{ for all } t \ge t_0. \tag{3.30}$$

Equation (3.30) is a special case of (3.11) with $F(t,x) = \Sigma(t)x$. Let the perturbation Σ be sufficiently small such that

$$\sup_{t\ge t_0} \|\Sigma(t)\| < \left(\sup_{t\ge t_0} \|HQ_\sigma G^{-1}(t)\|\right)^{-1}. \tag{3.31}$$

By using (3.31) and the relation

$$(I - \Sigma HQ_\sigma G^{-1})^{-1} G_\Sigma = G,$$

where $G_\Sigma := E_\sigma - (\bar{A} + \Sigma)HQ_\sigma$, it follows directly that G_Σ is invertible if only if so is G. This means that Equation (3.1) is of index-1 if and only if Equation (3.30) is, too. By using the same argument as before, we can solve Equation (3.30). Indeed, since the function $HQ_\sigma G^{-1}\Sigma(t)x$ is Lipschitz continuous with Lipschitz constant

$$\gamma_t = \|HQ_\sigma G^{-1}\Sigma(t)\| < 1,$$

the function g_t defined by (3.14) can be rewritten as follows

$$g_t(u) = (I - HQ_\sigma G^{-1}\Sigma(t))^{-1} HP_\sigma G^{-1}(\bar{A} + \Sigma)(t)u.$$

Then the solution of (3.30) is

$$x(t,s) = u(t,s) + g_t(u(t,s)),$$

where $u(t,s)$ is the solution of the IVP

$$\begin{cases} u^\Delta = (P^\Delta + P_\sigma G^{-1}\bar{A})u + P_\sigma G^{-1}\Sigma(u + g_t(u)), \\ u(s,s) = P(s)x_0. \end{cases}$$

Theorem 3.26. *Let Assumption* 3.2 *holds. Then, for any* $\varepsilon > 0$ *there exists a number* $\delta = \delta(\varepsilon) > 0$*, such that the inequality*

$$\limsup_{t\to\infty} \|\Sigma(t)\| \le \delta$$

implies

$$\kappa_{\mathcal{B}}(E, A + \Sigma) \le \kappa_{\mathcal{B}}(E, A) + \varepsilon.$$

Proof. Denote by $\Psi(t,s)$ the Cauchy operator of Equation (3.30). From the relation (3.8) and the variation of constants formula (3.10), it follows that

$$\Psi(t,s) = \Phi(t,s)P(s) + \int_s^t \Phi(t,\sigma(\tau))P_\sigma G^{-1}\Sigma\Psi(t,\tau)\Delta\tau + HQ_\sigma G^{-1}\Sigma\Psi(t,s).$$

Hence,

$$\begin{aligned} \Psi(t,s) = {} & (I - HQ_\sigma G^{-1}\Sigma(t))^{-1} \\ & \times \left(\Phi(t,s)P(s) + \int_s^t \Phi(t,\sigma(\tau))P_\sigma G^{-1}\Sigma\Psi(t,\tau)\Delta\tau \right). \end{aligned} \tag{3.32}$$

Since $HQ_\sigma G^{-1}$ is bounded on $\mathbb{T}_{t_0}$, we can choose $\delta_0 > 0$ such that

$$\|(I - HQ_\sigma G^{-1}\Sigma(t))^{-1}\| \le 2, \tag{3.33}$$

for all $t \in \mathbb{T}_a$, if $\sup_{t>t_0} \|\Sigma(t)\| < \delta_0$. For any $\varepsilon > 0$, put $\alpha = \kappa_{\mathcal{B}} + \frac{\varepsilon}{2}$. By Definition 3.17, there exists a number $M_\alpha > 0$ such that

$$\|\Phi(t,s)\| \le M_\alpha e_\alpha(t,s), \tag{3.34}$$

for all $t \geq s \geq t_0$. Combining (3.32), (3.33) and (3.34), we obtain

$$\|\Psi(t,s)\| \leq 2M_\alpha e_\alpha(t,s) + 2K_3M_\alpha \int_s^t e_\alpha(t,\sigma(\tau))\|\Sigma(\tau)\|\|\Psi(\tau,s)\|\Delta\tau,$$

or equivalently,

$$\begin{aligned} &e_{\ominus\alpha}(t,s)\|\Psi(t,s)\| \\ &\quad\leq 2M_\alpha + 2K_3M_\alpha \int_s^t \frac{1}{1+\mu(\tau)\alpha} e_{\ominus\alpha}(\tau,s)\|\Sigma(\tau)\|\|\Psi(\tau,s)\|\Delta\tau \\ &\quad\leq 2M_\alpha + 2K_3M_\alpha \sup_{t>t_0}\|\Sigma(t)\| \int_s^t \frac{1}{1+\mu(\tau)\alpha} e_{\ominus\alpha}(\tau,s)\|\Psi(\tau,s)\|\Delta\tau. \end{aligned}$$

Let $\delta = \min\{\delta_0, (2K_3M_\alpha)^{-1}\varepsilon\}$. If $\sup_{t>t_0}\|\Sigma(t)\| < \delta$, we then have

$$e_{\ominus\alpha}(t,s)\|\Psi)(t,s)\| \leq 2M_\alpha + \varepsilon \int_s^t \frac{1}{1+\mu(\tau)\alpha} e_{\ominus\alpha}(\tau,s)\|\Psi(\tau,s)\|\Delta\tau.$$

By applying Gronwall's inequality, we obtain

$$e_{\ominus\alpha}(t,s)\|\Psi(t,s)\| \leq 2M_\alpha e_{\frac{\varepsilon}{1+\mu(t)\alpha}}(t,s),$$

for all $t \geq s \geq t_0$. Thus,

$$\|\Psi(t,s)\| \leq 2M_\alpha e_{\frac{\varepsilon}{1+\mu(t)\alpha}}(t,s)e_\alpha(t,s) = 2M_\alpha e_{\alpha\oplus\frac{\varepsilon}{1+\mu(t)\alpha}}(t,s) = 2M_\alpha e_{\alpha+\varepsilon}(t,s)$$

for all $t \geq s \geq t_0$. This means that

$$\kappa_{\mathcal{B}}(E, A+\Sigma) \leq \kappa_{\mathcal{B}}(E,A) + \varepsilon.$$

The proof is complete. □

We now consider the equation

$$E_\sigma(t)x^\Delta(t) = A(t)x(t), \text{ for all } t \geq t_0,$$

subject to two-side perturbations of the form

$$\big(E_\sigma(t) + F_\sigma(t)\big)x^\Delta(t) = \big(A(t) + \Sigma(t)\big)x(t), \text{ for all } t \geq t_0, \tag{3.35}$$

where $F_\sigma(t)$ and $\Sigma(t)$ are the perturbation matrices. It is already known (e.g. [7, 51]) that for differential-algebraic equations and implicit difference equations, it is necessary to restrict the structure of perturbation in the matrix $E_\sigma(t)$ in order to get a meaningful problem of robust stability. Notice that under infinitesimally small perturbations, the solvability and/or the stability may be lost, which usually happens due to change in the regularity/index of the equation. Therefore, we assume that $F_\sigma(t)$ is an allowable structured perturbation, i.e.,

$$\ker(E_\sigma(t) + F_\sigma(t)) = \ker E_\sigma(t),$$

for all $t \in \mathbb{T}$. Let us define

$$G_F := E_\sigma + F_\sigma - (\bar{A} + \Sigma)HQ_\sigma = G_\Sigma + F_\sigma.$$

This implies that if F and Σ are small enough then Equation (3.35) is of index-1 and

$$G_F^{-1} = G_\Sigma^{-1} - G_\Sigma^{-1}F_\sigma(G_\Sigma + F_\sigma)^{-1}.$$

Multiplying both sides of (3.35) by $P_\sigma G_F^{-1}$ and $Q_\sigma G_F^{-1}$, we decouple Equation (3.35) into the system

$$\begin{aligned}(Px)^\Delta &= \Big(P^\Delta + P_\sigma\big(G_\Sigma^{-1} - G_\Sigma^{-1}F_\sigma(G_\Sigma + F_\sigma)^{-1}\big)\big(\bar{A} + \Sigma + F_\sigma P^\Delta\big)\Big)Px,\\ Qx &= HQ_\sigma\big(G_\Sigma^{-1} - G_\Sigma^{-1}F_\sigma(G_\Sigma + F_\sigma)^{-1}\big)\big(\bar{A} + \Sigma + F_\sigma P^\Delta\big)Px.\end{aligned}$$

Let us define

$$\Gamma := F_\sigma P^\Delta P - F_\sigma(G_\Sigma + F_\sigma)^{-1}(\bar{A} + \Sigma + F_\sigma P^\Delta)P, \quad \bar{\Sigma} := \Sigma + \Gamma.$$

Then, the above system becomes

$$\begin{aligned}(Px)^\Delta &= (P^\Delta + P_\sigma G_\Sigma^{-1}(\bar{A} + \Sigma))Px + P_\sigma G_\Sigma^{-1}\Gamma x,\\ Qx &= HQ_\sigma G_\Sigma^{-1}(\bar{A} + \Sigma)Px + HQ_\sigma G_\Sigma^{-1}\Gamma x.\end{aligned}$$

Therefore, Equation (3.35) is equivalent to

$$E_\sigma(t)x^\Delta(t) = \big(A(t) + \bar{\Sigma}(t)\big)x(t), \text{ for all } t \geq t_0.$$

From above argument and Theorem 3.26, we get the following theorem.

Theorem 3.27. *Let Assumption* 3.2 *holds. Then, for any* $\varepsilon > 0$ *there exists a number* $\delta = \delta(\varepsilon) > 0$ *such that the inequality*

$$\limsup_{t\to\infty} \|\bar{\Sigma}(t)\| \leq \delta$$

implies

$$\kappa_{\mathcal{B}}(E + F, A + \bar{\Sigma}) \leq \kappa_{\mathcal{B}}(E, A) + \varepsilon.$$

Conclusions of Chapter 3. In this chapter, we have investigated the robust stability, the Bohl exponent and the Bohl-Perron type theorem for linear time-varying implicit dynamic equations and derived some results as follows:

1. Establishing the solution formula (3.10) for linear time-varying implicit dynamic equations of the form $E_\sigma(t)x^\Delta = A(t)x + f(t)$;
2. Deriving some characterization for the robust stability of the IDEs with Lipschitz perturbations in Theorem 3.10. The Bohl-Perron type stability theorem is also extended for these equation in Theorem 3.14;
3. Defining the Bohl exponent of the equation $E_\sigma(t)x^\Delta = A(t)x$ on time scales and presenting the relation among the exponential stability, the Bohl exponent and solutions of the Cauchy problem, Theorem 3.23. The robustness of Bohl exponent when the dynamic equations under perturbations acting on both sides is investigated in Theorem 3.26 and Theorem 3.27.

We have extended and unified many previous results for the robust stability of time-varying systems: ordinary differential and difference equations, differential-algebraic and implicit difference equations, such as Theorem 3.14. There are many open problems that need to study in the future, e.g., the relation between Bohl exponent and robust stability for implicit dynamic equations under non-linear perturbations.

CHAPTER 4

STABILITY RADIUS FOR IMPLICIT DYNAMIC EQUATIONS

In this chapter, we will study the stability radius of the linear time-varying IDEs on time scales of the form

$$E_\sigma(t)x^\Delta(t) = A(t)x(t) + f(t), \text{ for all } t \ge t_0, \tag{4.1}$$

where $E_\sigma(\cdot) \in L_\infty^{\mathrm{loc}}(\mathbb{T}, \mathbb{K}^{n\times n})$ is supposed to be singular for all $t \in \mathbb{T}$, $t \ge 0$. The matrix $A(\cdot) \in L_\infty^{\mathrm{loc}}(\mathbb{T}, \mathbb{K}^{n\times n})$, and $\ker A(\cdot)$ is absolutely continuous. The corresponding homogeneous equation is

$$E_\sigma(t)x^\Delta(t) = A(t)x(t), \text{ for all } t \ge t_0. \tag{4.2}$$

The content of Chapter 4 is based on the paper No.2 in list of the author's scientific works.

Let X, Y be two finite-dimensional vector spaces. For any real number p, $1 \le p < \infty$ and $s < t$, $s, t \in \mathbb{T}$, we denote by $L_p([s,t];X)$ the space of measurable functions f defined on the closed interval $[s,t]$ equipped with the norm

$$\|f\|_{L_p([s,t];X)} := \left(\int_s^t \|f(\tau)\|^p \,\Delta\tau\right)^{\frac{1}{p}} < \infty,$$

and by $L_\infty([s,t];X)$ the space of measurable and essentially bounded functions f on $[s,t]$ equipped with the norm

$$\|f\|_{L_\infty([s,t];X)} := \Delta\text{-}\operatorname{esssup}_{\tau\in[s,t]} \|f(\tau)\|.$$

In short, we write $\|f\|_p$ for $\|f\|_{L_p([s,t];X)}$ and $\|f\|_\infty$ for $\|f\|_{L_\infty([s,t];X)}$ if there is not any confusion.

We also consider the spaces $L_p^{\mathrm{loc}}(\mathbb{T}_a; X)$ and $L_\infty^{\mathrm{loc}}(\mathbb{T}_a; X)$, which contain all functions f, such that the restriction of the function f on the closed interval $[s,t]$, denoted by $f|_{[s,t]}$, are in $L_p([s,t]; X)$ and $f|_{[s,t]} \in L_\infty([s,t]; X)$, respectively, for every $s, t \in \mathbb{T}_a$, $a \le s < t < \infty$.

For any $\tau \ge a, \tau \in \mathbb{T}$, the operator of truncation π_τ on the space $L_p(\mathbb{T}_a; X)$ is defined by

$$\pi_\tau(u)(t) := \begin{cases} u(t), & \text{if } t \in [a, \tau], \\ 0, & \text{if } t > \tau. \end{cases}$$

Denote by $\mathcal{L}(L_p(\mathbb{T}_a; X), L_p(\mathbb{T}_a; Y))$ the Banach space of linear bounded operators Σ from $L_p(\mathbb{T}_a; X)$ to $L_p(\mathbb{T}_a; Y)$ and the corresponding norm is defined by

$$\|\Sigma\| := \sup_{x \in L_p(\mathbb{T}_a;X),\ \|x\|=1} \|\Sigma x\|_{L_p(\mathbb{T}_a;Y)}.$$

The operator $\Sigma \in \mathcal{L}(L_p(\mathbb{T}_a; X), L_p(\mathbb{T}_a; Y))$ is called to be *causal* if it satisfies the equality

$$\pi_t \Sigma \pi_t = \pi_t \Sigma,$$

for every $t \in \mathbb{T}, t \ge a$. Sometimes, in this chapter, the time variable t will be omitted for brevity, if that does not cause any confusion.

4.1 Stability of IDEs under causal Perturbations

Firstly, we note that most of the contents presented in Section 3.1, Chapter 3 will be used to investigate the stability of linear time-varying IDEs

$$E_\sigma(t)x^\Delta(t) = A(t)x(t) + f(t),\ t \ge a, \tag{4.3}$$

and the corresponding homogeneous equation

$$E_\sigma(t)x^\Delta(t, t_0) = A(t)x(t, t_0), t \ge a \tag{4.4}$$

with the initial condition $P(t_0)(x(t_0, t_0) - x_0) = 0$.

Let $P(t), Q(t)$ be projectors defined in Section 3.1, Chapter 3, Equation (4.3) can be rewritten as

$$E_\sigma(t)(Px)^\Delta(t) = \bar{A}(t)x(t) + f(t),\ t \ge a, \tag{4.5}$$

with $\bar{A} := A + E_\sigma P^\Delta \in L_\infty^{\mathrm{loc}}(\mathbb{T}_a; \mathbb{K}^{n\times n})$.

From now on, we always suppose that the following assumptions hold.

Assumption 4.1. *The IDE* (4.4) *is of index-1 and uniformly exponentially stable in the sense that there exist numbers $M > 0$, $\omega > 0$ such that $-\omega$ is positively regressive and*

$$\|\Phi(t,s)\| \leq Me_{-\omega}(t,s),\ t \geq s,\ t,s \in \mathbb{T}_a.$$

Assumption 4.2. *There exists a bounded, smooth projector $Q(t)$ onto* $\ker E(t)$ *such that the terms $P_\sigma G^{-1}$ and $HQ_\sigma G^{-1}$ are essentially bounded on $\mathbb{T}_a$.*

Remark 4.1. i) Since $\Phi(t,t) = \widetilde{P}(t)$ for all $t \in \mathbb{T}_a$, it follows that $\widetilde{P}$, $\widetilde{Q}$ are bounded on $\mathbb{T}_a$ if the IDE (4.4) is uniformly exponentially stable.

ii) According to Lemma 3.3, the boundedness of terms $P_\sigma G^{-1}$, $HQ_\sigma G^{-1}$ is independent of the choice H, Q.

We consider Equation (4.4) subject to structured perturbations of the form

$$E_\sigma(t)x^\Delta(t) = A(t)x(t) + B(t)\Sigma\big(C(\cdot)x(\cdot)\big)(t),\ t \in \mathbb{T}_a, \tag{4.6}$$

where $B(\cdot) \in L_\infty(\mathbb{T}_a; \mathbb{K}^{n\times m})$ and $C(\cdot) \in L_\infty(\mathbb{T}_a; \mathbb{K}^{q\times n})$ are given matrices defining the structure of perturbations and

$$\Sigma : L_p(\mathbb{T}_a; \mathbb{K}^q) \to L_p(\mathbb{T}_a; \mathbb{K}^m)$$

is an unknown disturbance operator which supposed to be linear and causal. Therefore, with perturbation Σ, Equation (4.6) is an implicit functional DAE.

We now extend the concept of index-1 for Equation (4.6). Firstly, consider the linear operator $\widetilde{G}$ from $L_p^{\mathrm{loc}}(\mathbb{T}_a; \mathbb{K}^n)$ to $L_p^{\mathrm{loc}}(\mathbb{T}_a; \mathbb{K}^n)$ defined by

$$(\widetilde{G}u)(t) := (E_\sigma - \bar{A}HQ_\sigma)u(t) - B(t)\Sigma(CHQ_\sigma u(\cdot))(t),\ t \in \mathbb{T},$$

which can be written as follows $\widetilde{G} = (I - B\Sigma CHQ_\sigma G^{-1})G$.

The following lemma is taken from [2, Lemma 3.5.8]. Nevertheless, a more detailed and complete proof will be presented here.

Lemma 4.2. *Let $\mathbb{U} : X \to Y$, $\mathbb{V} : Y \to X$ be bounded linear operators in Banach spaces X, Y. Then the operator $I - \mathbb{U}\mathbb{V}$ is invertible if and only if $I - \mathbb{V}\mathbb{U}$ is invertible. Furthermore,*

$$(I - \mathbb{V}\mathbb{U})^{-1} = I + \mathbb{V}(I - \mathbb{U}\mathbb{V})^{-1}\mathbb{U}.$$

Proof. Suppose that $I - \mathbb{U}\mathbb{V}$ is invertible. By direct calculation, it can be seen directly that

$$(I + \mathbb{V}(I - \mathbb{U}\mathbb{V})^{-1}\mathbb{U})(I - \mathbb{V}\mathbb{U}) = (I + \mathbb{V}\mathbb{U})(I + \mathbb{V}(I - \mathbb{U}\mathbb{V})^{-1}\mathbb{U}) = I.$$

Thus, $I - \mathbb{V}\mathbb{U}$ is invertible and $(I - \mathbb{V}\mathbb{U})^{-1} = I + \mathbb{V}(I - \mathbb{U}\mathbb{V})^{-1}\mathbb{U}$. The proof is complete. $\square$

Applying Lemma 4.2 with two cases $\mathbb{U} = B, \mathbb{V} = \Sigma CHQ_\sigma G^{-1}$, and $\mathbb{U} = B\Sigma, \mathbb{V} = CHQ_\sigma G^{-1}$, it is clear that the operator $\widetilde{G}$ is invertible, if and only if the operators $I - \Sigma CHQ_\sigma G^{-1}B$, $I - CHQ_\sigma G^{-1}B\Sigma$ are invertible. Then, the concept of index-1 of Equation (4.6) is defined as follows.

Definition 4.3. The implicit functional DAE (4.6) is said to be of index-1, in the generalized sense, if for every $T > a$, the operator $\widetilde{G}$ restricted to $L_p([a, T]; \mathbb{K}^n)$ has the bounded inverse operator $\widetilde{G}^{-1}$.

For any $t_0 \in \mathbb{T}_a$, we set up the Cauchy problem for Equation (4.6)

$$\begin{cases} E_\sigma(t)x^\Delta(t) = A(t)x(t) + B(t)\Sigma\big(C(\cdot)[x(\cdot)]_{t_0}\big)(t), \\ P(t_0)(x(t_0) - x_0) = 0, \end{cases} \tag{4.7}$$

for all $t \in \mathbb{T}_{t_0}$, where $[x(t)]_{t_0} = \begin{cases} 0 & \text{if } t \in [a, t_0), \\ x(t) & \text{if } t \in [t_0, \infty). \end{cases}$

We say that the Cauchy problem (4.7) admits a mild solution if there exists $x(\cdot) \in L_p^{\text{loc}}(\mathbb{T}_{t_0}; \mathbb{K}^n)$ such that for all $t \geq t_0$, we have

$$\begin{aligned} x(t) = \Phi(t, t_0)P(t_0)x_0 + \int_{t_0}^{t} \Phi(t, \sigma(s))P_\sigma(s)G^{-1}(s)B(s)\Sigma\big(C(\cdot)[x(\cdot)]_{t_0}\big)(s)\Delta s \\ + H(t)Q_\sigma(t)G^{-1}(t)B(t)\Sigma\big(C(\cdot)[x(\cdot)]_{t_0}\big)(t). \end{aligned} \tag{4.8}$$

Now, we define operators

$$\begin{aligned} (\widehat{\mathbb{M}}_{t_0}u)(t) &:= \int_{t_0}^{t} \Phi(t, \sigma(s))P_\sigma(s)G^{-1}(s)B(s)u(s)\Delta s, \\ (\widetilde{\mathbb{M}}_{t_0}u)(t) &:= H(t)Q_\sigma(t)G^{-1}(t)B(t)u(t), \text{ and} \\ (\mathbb{M}_{t_0}u)(t) &:= (\widehat{\mathbb{M}}_{t_0}u)(t) + (\widetilde{\mathbb{M}}_{t_0}u)(t). \end{aligned}$$

We can directly see that $\mathbb{M}_{t_0}, \widehat{\mathbb{M}}_{t_0} \in \mathcal{L}(L_p([t_0,\infty);\mathbb{K}^m), L_p([t_0,\infty);\mathbb{K}^n))$ and there exists a constant $K_0 \geq 0$, such that

$$\|(\mathbb{M}_{t_0}u)(t)\| \leq K_0 \|u\|_{L_p([t_0,t];\mathbb{K}^m)}, \; t \geq t_0 \geq a, \; u|_{[t_0,t]} \in L_p([t_0,t];\mathbb{K}^m).$$

Denote by $x(t;t_0,x_0)$ the (mild) solution of the Cauchy problem (4.7). Then the formula (4.8) can be rewritten as

$$x(t;t_0,x_0) = \Phi(t,t_0)P(t_0)x_0 + \big(\mathbb{M}_{t_0}\Sigma(C(\cdot)[x(\cdot;t_0,x_0)]_{t_0})\big)(t).$$

The following theorem will show the existence and uniqueness of the solution to the IDE (4.7).

Theorem 4.4. *Assume that Equation* (4.7) *is of* index-1, *then it admits a unique mild solution $x(\cdot)$, where $P(\cdot)x(\cdot)$ is absolutely continuous with respect to Δ-measure. Furthermore, for an arbitrary number $T > t_0$, there exist positive constants $M_1 = M_1(T), M_2 = M_2(T)$ such that,*

$$\|P(t)x(t)\| \leq M_1 \|P(t_0)x_0\|, \quad \|x(t)\|_{L_p([t_0,t];\mathbb{K}^n)} \leq M_2\|P(t_0)x_0\|,$$

for all $t \in [t_0, T]$.

Proof. By using Equation (4.5) with $f = B\Sigma(C[x]_{t_0})$ and the variable changes $u = Px, v = Qx$, Equation (4.6) will be decoupled into the system

$$\begin{cases} u^\Delta = (P^\Delta + P_\sigma G^{-1}\bar{A})u + P_\sigma G^{-1}B\Sigma(C[u+v]_{t_0}), \\ v = HQ_\sigma G^{-1}\bar{A}u + HQ_\sigma G^{-1}B\Sigma(C[u+v]_{t_0}). \end{cases} \tag{4.9}$$

By the index-1 assumption and Lemma 4.2, $I - HQ_\sigma G^{-1}B\Sigma C$ is a bounded, invertible operator. Therefore, from the algebraic part of system (4.9), we get

$$[v]_{t_0} = (I - HQ_\sigma G^{-1}B\Sigma C)^{-1}HQ_\sigma G^{-1}(\bar{A} + B\Sigma C)[u]_{t_0} =: \mathbb{D}[u]_{t_0}. \tag{4.10}$$

Substituting $v = \mathbb{D}u$ into the delta-differential part gets

$$u^\Delta = (P^\Delta + P_\sigma G^{-1}\bar{A})u + P_\sigma G^{-1}B\Sigma C(I + \mathbb{D})[u]_{t_0} =: \mathbb{W}u. \tag{4.11}$$

We see that the operator $\mathbb{W}$ is linear, bounded and causal. Then, Equation (4.11) is equivalent to the integral equation

$$u(t) = u(t_0) + \int_{t_0}^{t} (\mathbb{W}u)(\tau)\Delta\tau.$$

By Picard approximation method, we can directly see that Equation (4.11) has a unique solution $u \in L_p^{\text{loc}}(\mathbb{T}_{t_0}; \mathbb{K}^n)$ with the initial condition

$$P(t_0)(x(t_0) - x_0) = 0, \; x_0 \in \mathbb{K}^n.$$

Then, we will get v from Equation (4.10) and obtain the solution $x = u + v \in L_p^{\text{loc}}(\mathbb{T}_{t_0}; \mathbb{K}^n)$. This unique solution can be defined by the variation of constants formula (4.8). In addition, the differential component $u = Px$ is absolutely continuous.

To prove the remainder part of Theorem 4.4, set $q := \|P\|_{L_\infty([t_0,t];\mathbb{K}^n)}$. According to the formula (4.8), we have

$$\begin{aligned}\|u(t)\| &= \|P(t)\Phi(t,t_0)P(t_0)x_0 + (P\mathbb{M}_{t_0}\Sigma(C(\cdot)[x(\cdot;t_0,x_0)]_{t_0})(t)\| \\ &\leq K_1\|P(t_0)x_0\| + K_0 q\|\Sigma\|\|C(\cdot)x(\cdot;t_0,x_0)\|_{L_p([t_0,t];\mathbb{K}^n)}.\end{aligned}$$

On the other hand,

$$\begin{aligned}\|C(\cdot)x(\cdot;t_0,x_0)\|^p_{L_p([t_0,t];\mathbb{K}^n)} &= \|C(\cdot)(I+\mathbb{D})u(\cdot)\|^p_{L_p([t_0,t];\mathbb{K}^n)} \\ &\leq K_2\|u(\cdot)\|^p_{L_p([t_0,t];\mathbb{K}^n)} \\ &\leq K_3\|P(t_0)x_0\|^p + K_4\int_{t_0}^t \|C(\cdot)x(\cdot;t_0,x_0)\|^p_{L_p([t_0,s];\mathbb{K}^n)}\Delta s.\end{aligned}$$

By applying Gronwall's inequality, we get

$$\begin{aligned}\|C(\cdot)x(\cdot;t_0,x_0)\|^p_{L_p([t_0,t];\mathbb{K}^n)} &\leq K_3\|P(t_0)x_0\|^p e_{K_4}(t_0,t) \\ &\leq K_3 e_{K_4}(t_0,T)\|P(t_0)x_0\|^p.\end{aligned}$$

and hence, by setting $K_5 = \left(K_3 e_{K_4}(t_0,T)\right)^{\frac{1}{p}}$ we have

$$\|C(\cdot)x(\cdot;t_0,x_0)\|_{L_p([t_0,t];\mathbb{K}^n)} \leq K_5\|P(t_0)x_0\|.$$

Thus, $\|P(t)x(t)\| = \|u(t)\| \leq M_1\|P(t_0)x_0\|$, where $M_1 := K_1 + K_0 q\|\Sigma\|K_5$. In addition, we have

$$\begin{aligned}&\|x(\cdot;t_0,x_0)\|_{L_p([t_0,t];\mathbb{K}^n)} \\ &\leq \|\Phi(\cdot,t_0)P(t_0)x_0 + (\mathbb{M}_{t_0}\Sigma(C(\cdot)[x(\cdot;t_0,x_0)]_{t_0}))(\cdot)\|_{L_p([t_0,t];K^n)} \\ &\leq \|\Phi(\cdot,t_0)P(t_0)x_0\|_{L_p([t_0,t];\mathbb{K}^n)} + \|(\mathbb{M}_{t_0}\Sigma(C(\cdot)[x(\cdot;t_0,x_0)]_{t_0}))(\cdot)\|_{L_p([t_0,t];K^n)} \\ &\leq K_6\left\|P(t_0)x_0\right\| + K_0\|\Sigma\|\|C(\cdot)x(\cdot;t_0,x_0)\|_{L_p([t_0,t];\mathbb{K}^n)} \\ &\leq M_2\left\|P(t_0)x_0\right\|,\end{aligned}$$

where $M_2 := K_6 + K_0\|\Sigma\|K_5$. The proof is complete. □

Remark 4.5. Let the operator $\Sigma \in \mathcal{L}(L_p(\mathbb{T}_a;\mathbb{K}^q), L_p(\mathbb{T}_a;\mathbb{K}^m))$ be causal, for all $t > a$ and $h \in L_p([a,t];\mathbb{K}^q)$. Then, by applying Theorem 4.4, we see that the function g defined by

$$g(s) := P(t)x(t;\sigma(s),h(s)),\ s \in [a,t],$$

belongs to $L_p([a,t];\mathbb{K}^n)$. Furthermore, set $y(t) := \int_s^t g(\tau)\Delta\tau$ then, by Theorem 1.25, we have

$$\begin{aligned} y^\Delta(t) &= P_\sigma(t)x(\sigma(t);\sigma(t),h(t)) + \int_s^t (P(t)x(t;\sigma(\tau),h(\tau)))^\Delta \Delta\tau \\ &= P_\sigma(t)h(t) + \int_s^t \left(\mathbb{W}P(\cdot)x(\cdot;\sigma(\tau),h(\tau))\right)(t)\Delta\tau \\ &= P_\sigma(t)h(t) + \mathbb{W}\left(\int_s^\cdot P(\cdot)x(\cdot;\sigma(\tau),h(\tau))\Delta\tau\right)(t) \\ &= P_\sigma(t)h(t) + (\mathbb{W}y)(t), \end{aligned}$$

where the operator $\mathbb{W}$ is defined in (4.11).

4.2 Stability Radius under Dynamic Perturbations

Let Assumptions 4.1, 4.2 hold. The trivial solution of Equation (4.6) is said to be *globally L_p-stable* if there exist positive constants M_3, M_4 such that

$$\begin{aligned} &\|P(t)x(t;t_0,x_0)\|_{\mathbb{K}^n} \le M_3 \|P(t_0)x_0\|_{\mathbb{K}^n}, \\ &\|x(t;t_0,x_0)\|_{L_p(T_{t_0},\mathbb{K}^n)} \le M_4 \|P(t_0)x_0\|_{\mathbb{K}^n} \end{aligned} \tag{4.12}$$

for all $t \ge t_0$, $x_0 \in \mathbb{K}^n$.

Next, we extend the definition of stability radius introduced in [38, 45, 68] for linear IDEs on time scales.

Definition 4.6. Let Assumptions 4.1, 4.2 hold. The complex (real) structured stability radius of Equation (4.2) subject to linear, dynamic and causal perturbations in Equation (4.6) is defined by

$$r_{\mathbb{K}}(E_\sigma, A; B, C; \mathbb{T}) := \inf \left\{ \begin{array}{l} \|\Sigma\|, \text{ the trivial solution of (4.6) is not} \\ \text{globally } L_p\text{-stable or (4.6) is not of index-1} \end{array} \right\}.$$

For every $t_0 \in \mathbb{T}_a$, we define the following operators

$$\widehat{\mathbb{L}}_{t_0} u := C(\cdot)\widehat{\mathbb{M}}_{t_0} u,\ \widetilde{\mathbb{L}}_{t_0} u := C(\cdot)\widetilde{\mathbb{M}}_{t_0} u, \text{ and } \mathbb{L}_{t_0} u := C(\cdot)\mathbb{M}_{t_0} u.$$

The operator $\mathbb{L}_{t_0}$ is called a *input-output operator* associated with the perturbed equation (4.6). It can be seen directly that $\mathbb{L}_{t_0}, \widehat{\mathbb{L}}_{t_0}$ are the operators from $L_p(\mathbb{T}_{t_0}; \mathbb{K}^m)$ to $L_p(\mathbb{T}_{t_0}; \mathbb{K}^q)$, and $\|\mathbb{L}_{t_0}\|$, $\|\widehat{\mathbb{L}}_{t_0}\|$ are the decreasing functions in t_0. Furthermore,

$$\|\widetilde{\mathbb{L}}_{t_0}\| = \Delta\text{-}\operatorname{esssup}_{t\geq t_0} \left\| CHQ_\sigma G^{-1}B \right\| \leq \|\mathbb{L}_{t_0}\|.$$

Note that $\|\mathbb{L}_t\|$ is decreasing in t. Therefore, there exists the limit

$$\|\mathbb{L}_\infty\| := \lim_{t\to\infty} \|\mathbb{L}_t\|.$$

Denote

$$\beta := \|\mathbb{L}_\infty\|^{-1}, \quad \gamma := \|\widetilde{\mathbb{L}}_a\|^{-1}, \tag{4.13}$$

with the convention $0^{-1} = +\infty$.

We say that, the causal operator $\mathbb{Q} \in \mathcal{L}(L_p(\mathbb{T}_a; \mathbb{K}^m), L_p(\mathbb{T}_a; \mathbb{K}^q))$ has *finite memory*, if there exists a function $\Psi : [a, \infty) \to [a, \infty)$ such that $\Psi(t) \geq t$ and $(I - \pi_{\Psi(t)})\mathbb{Q}\pi_t = 0$, for all $t \geq a$. The function Ψ is called a finite memory function associated with the operator $\mathbb{Q}$.

Since $\widetilde{\mathbb{L}}_{t_0} u$ defined by $(\widetilde{\mathbb{L}}_{t_0} u)(t) := C(t)H(t)Q_\sigma(t)G^{-1}(t)B(t)u(t)$ is a causal and finite memory operator, we can adopt the arguments in [45], and get the following lemma.

Lemma 4.7. *For any number $\varepsilon > 0$, there exists a causal operator*

$$\mathbb{Q}_\varepsilon \in \mathcal{L}\big(L_p(\mathbb{T}_a, \mathbb{K}^m), L_p(\mathbb{T}_a, \mathbb{K}^q)\big)$$

with finite memory such that $\|\mathbb{L}_a - \mathbb{Q}_\varepsilon\| < \varepsilon$.

To derive the main result in this section, we prove the following lemma.

Lemma 4.8. *Suppose that $\beta < \infty$ and $\alpha > \beta$, where β is defined in* (4.13). *Then, there exist an operator*

$$\Sigma \in \mathcal{L}\big(L_p(\mathbb{T}_a, \mathbb{K}^q), L_p(\mathbb{T}_a, \mathbb{K}^m)\big),$$

the functions $\tilde{y}, \tilde{z} \in L_p^{\text{loc}}(\mathbb{T}_a, \mathbb{K}^q)$ and a natural number $N_0 > 0$ such that

i) $\|\Sigma\| < \alpha$, Σ *is causal and has finite memory;*

ii) $\Sigma h(t) = 0$ *for every* $t \in [0, N_0]$ *and all* $h \in L_p(\mathbb{T}_a, \mathbb{K}^q)$*;*

iii) $\tilde{y} \in L_p^{\mathrm{loc}}(\mathbb{T}_a, \mathbb{K}^q) \setminus L_p(\mathbb{T}_a, \mathbb{K}^q)$ *and* $\operatorname{supp} \tilde{z} \subset [0, N_0]$*;*

iv) $(I - \mathbb{L}_a\Sigma)\tilde{y} = \tilde{z}$.

Proof. Set $\varepsilon := \frac{\alpha-\beta}{2\alpha\beta}$. By Lemma 4.7, there is a causal operator $\mathbb{Q}_\varepsilon$ with finite memory, $\mathbb{Q}_\varepsilon \in \mathcal{L}(L_p(\mathbb{T}_a, \mathbb{K}^m), L_p(\mathbb{T}_a, \mathbb{K}^q))$, such that

$$\|\mathbb{L}_a - \mathbb{Q}_\varepsilon\| < \varepsilon.$$

Set $\mathbb{Q}_{\varepsilon,t}u := \mathbb{Q}_\varepsilon[u]_t$. It is seen that $\|\mathbb{Q}_{\varepsilon,t} - \mathbb{L}_t\|$ is a decreasing operator in t. Therefore,

$$\|\mathbb{Q}_{\varepsilon,t}\| > \frac{1}{\beta} - \varepsilon,$$

for all $t \in \mathbb{T}_a$. Since

$$\|\mathbb{Q}_{\varepsilon,a}\| > \frac{1}{\beta} - \varepsilon,$$

there exists a function $\tilde{f}_0 \in L_p(\mathbb{T}_a, \mathbb{K}^q)$, such that

$$\|\mathbb{Q}_{\varepsilon,a}\tilde{f}_0\| > \frac{1}{\beta} - \varepsilon.$$

Therefore, we can choose an element $t_1 \in \mathbb{T}_a,$ such that

$$\|\pi_{t_1}\mathbb{Q}_{\varepsilon,a}\tilde{f}_0\| > \frac{1}{\beta} - \varepsilon.$$

Let Ψ be a finite memory function associated with $\mathbb{Q}_\varepsilon$. We define

$$f_0 := \frac{\pi_{t_1}\tilde{f}_0}{\|\pi_{t_1}\tilde{f}_0\|} \text{ and } N_0 := \Psi(t_1),$$

and get

$$\|f_0\| = 1,\ \|\mathbb{Q}_{\varepsilon,a}f_0\| > \frac{1}{\beta} - \varepsilon,\ \operatorname{supp} f_0 \subset [a, N_0],\ \operatorname{supp}\, \mathbb{Q}_{\varepsilon,a}f_0 \subset [a, N_0].$$

Set $K_1 = N_0 + 1$. We also have

$$\|\mathbb{Q}_{\varepsilon,K_1}\| > \frac{1}{\beta} - \varepsilon,$$

which implies that there are $N_1 > K_1$, the function $f_1 \in L_p(\mathbb{T}_a, \mathbb{K}^m)$ such that

$$\|f_1\| = 1,\ \|\mathbb{Q}_{\varepsilon,K_1} f_1\| > \frac{1}{\beta} - \varepsilon,\ \mathrm{supp}\, f_1 \subset [K_1, N_1],\ \mathrm{supp}\ \mathbb{Q}_{\varepsilon,K_1} f_1 \subset [K_1, N_1].$$

Continuing this way, we can set up sequences $\{f_n\} \subset L_p(\mathbb{T}_{t_0}, \mathbb{K}^m)$, $\{K_n\}$, and $\{N_n\}$ such that $K_n < N_n < K_{n+1}$ satisfying

$$\|f_n\| = 1, \|\mathbb{Q}_{\varepsilon,K_n} f_n\| > \frac{1}{\beta} - \varepsilon, \mathrm{supp}\, f_n \subset [K_n, N_n], \mathrm{supp}\ \mathbb{Q}_{\varepsilon,K_n} f_n \subset [K_n, N_n],$$

where $n = 0, 1, 2...$ and $K_0 := a$. We define the function $f := \sum_{n=0}^{\infty} f_n$, and see that $f,\ \mathbb{Q}_\varepsilon f \in L_p^{\mathrm{loc}}(\mathbb{T}_{t_0}, \mathbb{K}^q) \setminus L_p(\mathbb{T}_{t_0}, \mathbb{K}^q)$. Let Λ_n be a linear functional defined on $L_p([K_n, N_n]; \mathbb{K}^m)$ such that

$$\|\Lambda_n\| = 1,\ \Lambda_n(\mathbb{Q}_\varepsilon f_n|_{[K_n,N_n]}) = \|\mathbb{Q}_\varepsilon f_n\|.$$

We define the operator

$$\Sigma_\varepsilon(h) := \sum_{n=0}^{\infty} \frac{f_{n+1}}{\|\mathbb{Q}_\varepsilon f_n\|} \Lambda_n h|_{[K_n,N_n]}$$

for every $h \in L_p(\mathbb{T}_{t_0}, \mathbb{K}^q)$. We can directly see that the operator

$$\Sigma_\varepsilon \in \mathcal{L}(L_p(\mathbb{T}_{t_0}, \mathbb{K}^q), L_p(\mathbb{T}_{t_0}, \mathbb{K}^m))$$

is causal and has finite memory. Moreover,

$$\|\Sigma_\varepsilon\| < \frac{\beta}{1 - \varepsilon\beta}, \quad \Sigma_\varepsilon(h)(t) = 0$$

for $h \in L_p(\mathbb{T}_{t_0}, \mathbb{K}^q), t \in [a, N_0]$, and $\Sigma_\varepsilon(\mathbb{Q}_\varepsilon f_n) = f_{n+1}$. These properties of the operator Σ_ε imply that $(I - \Sigma_\varepsilon \mathbb{Q}_\varepsilon) f = f_0$. On the other hand, since

$$\|(\mathbb{Q}_\varepsilon - \mathbb{L}_a)\Sigma_\varepsilon\| < \frac{\varepsilon\beta}{1 - \varepsilon\beta} < 1,$$

the operator $I - (\mathbb{Q}_\varepsilon - \mathbb{L}_a)\Sigma_\varepsilon$ is invertible in $\mathcal{L}\big(L_p(\mathbb{T}_{t_0}, \mathbb{K}^q), L_p(\mathbb{T}_{t_0}, \mathbb{K}^m)\big)$. We now define

$$\Sigma := \Sigma_\varepsilon[I - (\mathbb{Q}_\varepsilon - \mathbb{L}_a)\Sigma_\varepsilon]^{-1} \in \mathcal{L}\big(L_p(\mathbb{T}_{t_0}, \mathbb{K}^q), L_p(\mathbb{T}_{t_0}, \mathbb{K}^m)\big).$$

Since the operator Σ_ε is causal and has finite memory, it is clear that Σ is causal and has finite memory as well. Moreover, we have

$$\Sigma = \sum_{k=0}^{\infty} \Sigma_\varepsilon[(\mathbb{Q}_\varepsilon - \mathbb{L}_a)\Sigma_\varepsilon]^k,$$

it implies that $\Sigma(h)(t) = 0$ for all $t \in [a, N_0], h \in L_p(\mathbb{T}_{t_0}, \mathbb{K}^q)$ and

$$\|\Sigma\| \le \sum_{k=0}^{\infty} \|\Sigma_\varepsilon\| \|(\mathbb{Q}_\varepsilon - \mathbb{L}_a)\Sigma_\varepsilon\|^k < \frac{\beta}{1-\varepsilon\beta} \sum_{k=0}^{\infty} \left(\frac{\varepsilon\beta}{1-\varepsilon\beta}\right)^k = \frac{\beta}{1-2\varepsilon\beta} = \alpha.$$

Let us define

$$\tilde{y} := (I - (\mathbb{Q}_\varepsilon - \mathbb{L}_a)\Sigma_\varepsilon)\mathbb{Q}_\varepsilon f, \quad \tilde{z} := \mathbb{Q}_\varepsilon f_0.$$

Since

$$\mathbb{Q}_\varepsilon f \in L_p^{\mathrm{loc}}(\mathbb{T}_{t_0}, \mathbb{K}^q) \setminus L_p(\mathbb{T}_{t_0}, \mathbb{K}^q)$$

and $I - (\mathbb{Q}_\varepsilon - \mathbb{L}_a)\Sigma_\varepsilon$ is invertible, we get $\tilde{y} \in L_p^{\mathrm{loc}}(\mathbb{T}_{t_0}, \mathbb{K}^q) \setminus L_p(\mathbb{T}_{t_0}, \mathbb{K}^q)$. Moreover, $\operatorname{supp} \tilde{z} \subset \operatorname{supp} f_0 \subset [a, N_0]$ and

$$(I - \mathbb{L}_a\Sigma)\tilde{y} = (I - \mathbb{Q}_\varepsilon\Sigma_\varepsilon)\mathbb{Q}_\varepsilon f = \mathbb{Q}_\varepsilon f_0 = \tilde{z}.$$

The proof is complete. □

We are now in position to derive the main result in this section.

Theorem 4.9. *Let Assumptions* 4.1, 4.2 *hold. Then*

$$r_{\mathbb{K}}(E_\sigma, A; B, C; \mathbb{T}) = \min\{\beta, \gamma\}. \tag{4.14}$$

where β, γ *are defined in* (4.13).

Proof. The proof is divided into three steps.

Step 1. We will prove that $r_{\mathbb{K}}(E_\sigma, A; B, C; \mathbb{T}) \ge \min\{\beta, \gamma\}$.

Consider the first case where $\beta < \infty, \gamma < \infty$. Assume that Σ is a linear and causal perturbation with $\|\Sigma\| < \min\{\beta, \gamma\}$. Then, we have

$$\|\Sigma\| < \gamma = \|\widetilde{\mathbb{L}}_a\|^{-1} = \left(\operatorname{esssup}_{t \ge a} \|CHQ_\sigma G^{-1}B\|\right)^{-1}.$$

Therefore,

$$\|CHQ_\sigma G^{-1}B\Sigma\| < 1, \text{ almost } t \in \mathbb{T}_a,$$

which implies that the matrix $I - CHQ_\sigma G^{-1}B\Sigma$ is invertible, and so, by Lemma 4.2 and Definition 4.3, it is clear that Equation (4.6) is of index-1. Consequently, it admits a unique mild solution $x(t; t_0, x_0)$ for all $t_0 \ge a$, $x_0 \in \mathbb{K}^n$. On the other hand,

$$\|\Sigma\| < \beta = \lim_{t \to \infty} \|\mathbb{L}_t\|^{-1},$$

which implies that there exits a number $T > a$ such that

$$\|\Sigma\| \|\mathbb{L}_T\| < 1.$$

From the formula (4.8), it follows that

$$C(t)x(t;t_0,x_0) = C(t)\Phi(t,T)P(T)x(T;t_0,x_0) + \mathbb{L}_T(\Sigma(C(\cdot)[x(\cdot)]_{t_0}))(t)$$

for all $t \geq T$. Therefore, by Assumption 4.1, we have

$$\begin{aligned}
\|C(\cdot)x(\cdot;t_0,x_0)\|_{L_p(\mathbb{T}_T,\mathbb{K}^q)} &\leq \|C(\cdot)\Phi(\cdot,T)P(T)x(T;t_0,x_0)\|_{L_p(\mathbb{T}_T,\mathbb{K}^q)} \\
&\quad + \|\mathbb{L}_T\| \|\Sigma(C(\cdot)[x(\cdot;t_0,x_0)]_{t_0})\|_{L_p(\mathbb{T}_T,\mathbb{K}^q)} \\
&\leq M_5 \|P(t_0)x_0\| + \|\mathbb{L}_T\| \|\Sigma\| \|C(\cdot)[x(\cdot;t_0,x_0)]_{t_0}\|_{L_p(\mathbb{T}_{t_0},\mathbb{K}^q)} \\
&= M_5 \|P(t_0)x_0\| + \|\mathbb{L}_T\| \|\Sigma\| \|C(\cdot)x(\cdot;t_0,x_0)\|_{L_p(\mathbb{T}_{t_0},\mathbb{K}^q)}.
\end{aligned}$$

Hence,

$$\begin{aligned}
&\|C(\cdot)x(\cdot;t_0,x_0)\|_{L_p(\mathbb{T}_{t_0},\mathbb{K}^q)} \\
&\quad \leq \|C(\cdot)x(\cdot;t_0,x_0)\|_{L_p([t_0,T],\mathbb{K}^q)} + \|C(\cdot)x(\cdot;t_0,x_0)\|_{L_p(\mathbb{T}_T,\mathbb{K}^q)} \\
&\quad \leq (M_1 + M_5) \|P(t_0)x_0\| + \|\mathbb{L}_T\| \|\Sigma\| \|C(\cdot)x(\cdot;t_0,x_0)\|_{L_p(\mathbb{T}_{t_0},\mathbb{K}^q)}.
\end{aligned}$$

This proves that

$$\|C(\cdot)x(\cdot;t_0,x_0)\|_{L_p(\mathbb{T}_{t_0},\mathbb{K}^q)} \leq \frac{(M_1 + M_5) \|P(t_0)x_0\|}{1 - \|\mathbb{L}_T\| \|\Sigma\|}.$$

Note that by Assumption 4.1, the constants $K_i, i = 0, 1, \ldots, 6$ in the proof of Theorem 4.4 do not depend on t. Thus, similar to the second part of this proof, (4.12) holds and the perturbed equation (4.6) is globally L_p-stable. In the second case $\beta = \infty$ or $\gamma = \infty$, the above arguments still hold true for $\|\Sigma\| < \min\{\beta, \gamma\}$.

Step 2. We will prove that $r_{\mathbb{K}}(E_\sigma, A; B, C; \mathbb{T}) \leq \gamma$.

Without loss of generality, we assume that $\gamma < \infty$. Due to the definition of essential supremum, for any $\varepsilon > 0$, there exists a closed set $J \subseteq \mathbb{T}_a$ with Δ-positive measure, such that

$$\|C(t)H(t)Q_\sigma(t)G^{-1}(t)B(t)\| \geq (\gamma + \varepsilon)^{-1}, \text{ for all } t \in J.$$

We consider the linear mapping $\Gamma : L_\infty(J, \mathbb{K}^q) \to L_\infty(J, \mathbb{K}^q)$ defined by

$$(\Gamma u)(t) := C(t)H(t)Q_\sigma(t)G^{-1}(t)B(t)u(t), \text{ for all } t \in J.$$

It is clear that

$$\|\Gamma\| = \Delta\text{-}\operatorname{esssup}_{t\in J} \|C(t)H(t)Q_\sigma(t)G^{-1}(t)B(t)\| > (\gamma+\varepsilon)^{-1}.$$

By Kuratowski and Ryll-Nardzewski Theorem [70, Theorem 5.2.1], we can find a measurable function v defined in J with condition $\|v(t)\| = 1$, for all $t \in J$, such that

$$\|C(t)H(t)Q_\sigma(t)G^{-1}(t)B(t)v(t)\| = \|C(t)H(t)Q_\sigma(t)G^{-1}(t)B(t)\|,$$

for all $t \in J$. This means that $\|\Gamma v\| = \|\Gamma\|$. By using Hahn-Banach Theorem we can find a linear functional Λ on $L_\infty(J, \mathbb{K}^q)$, such that $\|\Lambda\| = 1$ and $\Lambda(\Gamma v) = \|\Gamma v\| = \|\Gamma\|$. We define

$$(\Sigma_0 u)(t) := \frac{v(t)\Lambda(u)}{\|\Gamma\|},$$

for all $t \in J,\ u \in L_\infty(J, \mathbb{K}^m)$. It is clear that

$$\left((I - \Sigma_0 C(\cdot)H(\cdot)Q_\sigma(\cdot)G^{-1}(\cdot)B(\cdot))v\right)(t) = v(t) - \frac{v(t)\Lambda(\Gamma v)(t)}{\|\Gamma\|} = 0.$$

Thus, $I - \Sigma_0 CHQ_\sigma G^{-1}B$ is not invertible in J. Let Σ be a causal perturbation operator defined by

$$(\Sigma u)(t) := \begin{cases} \Sigma_0 u(t), & \text{if } t \in J \\ 0, & \text{if } t \notin J. \end{cases}$$

It is clear that $\|\Sigma\| = \|\Sigma_0\| < \gamma + \varepsilon$ and $I - \Sigma_0 CHQ_\sigma G^{-1}B$ is not invertible, which implies that Equation (4.6) is not of index-1 which is a contradiction. Since $\varepsilon > 0$ is arbitrary small, we get

$$r_{\mathbb{K}}(E_\sigma, A; B, C; \mathbb{T}) \le \gamma.$$

Step 3. We will prove that $r_{\mathbb{K}}(E_\sigma, A; B, C; \mathbb{T}) \le \beta$.

Without loss of generality, we assume that $\beta < \infty$. Indeed, if $\beta \ge \gamma$, then this is evident by Step 2. Therefore, we can assume that $\beta < \gamma$. On the contrary, suppose that

$$\beta < r_{\mathbb{K}}(E_\sigma, A; B, C; \mathbb{T}) = \alpha < \gamma.$$

Then, we can find a number $N_0 > a$, a causal perturbation operator

$$\Sigma \in \mathcal{L}(L_p(\mathbb{T}_a, \mathbb{K}^q), L_p(\mathbb{T}_a, \mathbb{K}^m)),$$

and the functions $\tilde{y}, \tilde{z} \in L_p^{\text{loc}}(\mathbb{T}_a, \mathbb{K}^q)$ which satisfy the conditions in Lemma 4.8. Define

$$f := \tilde{y}|_{[a,N_0]} \text{ and } y := \tilde{y}|_{[N_0,\infty)},$$

we have that

$$\begin{aligned} y(t) = \tilde{y}(t) &= (\mathbb{L}_0\Sigma\tilde{y})(t) + \tilde{z}(t) = (\mathbb{L}_0\Sigma\tilde{y})(t) \\ &= (\mathbb{L}_{N_0}\Sigma\tilde{y})(t) + (\mathbb{L}_0\pi_{N_0}\Sigma\tilde{y})(t) \\ &= \big(\mathbb{L}_{N_0}\Sigma(\pi_{N_0}\tilde{y} + [\tilde{y}]_{N_0})\big)(t) + \big(\mathbb{L}_0\pi_{N_0}\Sigma\pi_{N_0}\tilde{y}\big)(t) \\ &= (\mathbb{L}_{N_0}\Sigma)(t) + (\mathbb{L}_{N_0}\Sigma y)(t). \end{aligned}$$

for all $t \geq N_0$. Let

$$x_y(t) := (\mathbb{M}_{N_0}\Sigma f)(t) + (\mathbb{M}_{N_0}\Sigma y)(t),\ t \geq N_0. \tag{4.15}$$

It is clear that

$$C(\cdot)x_y(\cdot) = (\mathbb{L}_{N_0}\Sigma f)(\cdot) + (\mathbb{L}_{N_0}\Sigma y)(\cdot) = y(\cdot) \in L_p^{\text{loc}}(\mathbb{T}_{t_0}, \mathbb{K}^q) \setminus L_p(\mathbb{T}_{t_0}, \mathbb{K}^q).$$

Thus, $x_y(\cdot)$ is a solution of the equation

$$E_\sigma(t)x_y^\Delta(t) = A(t)x_y(t) + B(t)\Sigma\big(C(\cdot)[x_y(\cdot)]_{N_0}\big)(t) + B(t)\Sigma(f)(t),$$

with the initial condition $P(N_0)x_y(N_0) = 0$. Similar to the decomposition into the equations (4.9) and (4.10), we see that $P(t)x_y(t)$ is the unique solution of the equation $(Px_y)^\Delta = \mathbb{W}Px_y + P_\sigma h$, where $\mathbb{W}$ is defined in (4.11) and h is defined by

$$h := G^{-1}B\Sigma C(I - HQ_\sigma G^{-1}B\Sigma C)^{-1}HQ_\sigma G^{-1}B\Sigma f + G^{-1}B\Sigma f.$$

By Remark 4.5, we have

$$P(t)x_y(t) = \int_{N_0}^t P(t)x(t;\sigma(s), h(s))\Delta s.$$

It is clear that the assumption $C(\cdot) \in L_\infty(\mathbb{T}_{t_0}, \mathbb{K}^{q\times n})$ implies

$$x_y(\cdot) \in L_p^{\text{loc}}(\mathbb{T}_{t_0}, \mathbb{K}^n) \setminus L_p(\mathbb{T}_{t_0}, \mathbb{K}^n).$$

Since Σ as well as $(I - CHQ_\sigma G^{-1}B\Sigma)^{-1} = \sum_{k=0}^\infty (CHQ_\sigma G^{-1}B\Sigma)^k$ are the finite memory operators, so is $(I - HQ_\sigma G^{-1}B\Sigma C)^{-1}$, by Lemma 4.2. Furthermore, since f has a compact support, so does h.

Now, we suppose that the trivial solution of Equation (4.6) is globally L_p-stable. This implies that $Px_y(\cdot) \in L_p(\mathbb{T}_{t_0};\mathbb{K}^n)$. To this end, we use the estimate

$$\begin{aligned}
\|Px_y(t)\|_{L_p([N_0,\infty);\mathbb{K}^n)} &= \left[\int_{N_0}^{\infty}\left\|\int_{N_0}^{t} P(t)x(t;\sigma(s),h(s))\Delta s\right\|^p \Delta t\right]^{\frac{1}{p}} \\
&\leq \left[\int_{N_0}^{\infty}\left(\int_{N_0}^{t} \|P(t)x(t;\sigma(s),h(s))\|\,\Delta s\right)^p \Delta t\right]^{\frac{1}{p}} \\
&\leq \int_{N_0}^{\infty}\left(\int_{s}^{\infty} \|P(t)x(t;\sigma(s),h(s))\|^p\,\Delta t\right)^{\frac{1}{p}} \Delta s \\
&\leq M_3\int_{N_0}^{\infty}\|h(s)\|\,\Delta s < +\infty.
\end{aligned}$$

Consequently, both $CPx_y(\cdot)$ and $CQx_y(\cdot)$ belong to $L_p(\mathbb{T}_{t_0},\mathbb{K}^q)$, which is contradicted to the fact that

$$Cx_y(\cdot) \in L_p^{\mathrm{loc}}(\mathbb{T}_{t_0},\mathbb{K}^q) \setminus L_p(\mathbb{T}_{t_0},\mathbb{K}^q).$$

Thus, the trivial solution of Equation (4.6) is not globally L_p-stable. The proof is complete. □

Remark 4.10. In case $\mathbb{T} = \mathbb{R}$, (4.14) gives the stability radius formula in [24, Theorem 2], and in case $\mathbb{T} = \mathbb{Z}$ we obtain the stability radius formula in [69, Theorem 4.6]. However, the above proof has some modifications using different techniques and it is essentially simpler than the proofs in [24, 69].

Remark 4.11. In case $\mathbb{T} = \mathbb{R}$ and $E = I$, (4.14) gives the stability radius formula in [45, Theorem 4.1]. However, since the operator of the left shift may not exist on an arbitrary time scale, we have derived Lemma 4.8 in order to illustrate that causal perturbations may destroy global L_p-stability. This fact is different from [45].

Example 4.12. Consider Equation (4.2) with

$$E(t) = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},\ A(t) = \begin{bmatrix} p(t) & p(t) & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

on time scale $\mathbb{T} = \bigcup_{k=0}^{\infty}\{3k\} \bigcup_{k=0}^{\infty}[3k+1, 3k+2]$, where

$$p(t) = \begin{cases} -\frac{1}{2} & \text{if } t = 3k, \\ -\frac{1}{4} & \text{if } t \in [3k+1, 3k+2]. \end{cases} \tag{4.16}$$

In this case, we can choose and compute that

$$P = \widetilde{P} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 \end{bmatrix}, H = I, G^{-1} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

Simple calculations yield that the transition matrices of the equation (E, A) are given by

$$\Phi_0(t,s) = \frac{1}{2} \begin{bmatrix} e_p(t,s)+1 & e_p(t,s)-1 & 0 \\ e_p(t,s)-1 & e_p(t,s)+1 & 0 \\ 0 & 0 & 2 \end{bmatrix},$$
$$\Phi(t,s) = \frac{1}{2} \begin{bmatrix} e_p(t,s) & e_p(t,s) & 0 \\ e_p(t,s) & e_p(t,s) & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Assume that $B = C = I$ are the matrices defining the structure of perturbation in the perturbed equation (4.6). Then, we have

$$(\mathbb{L}_{t_0}u)(t) = \left(\int_{t_0}^{t} e_p(t,\sigma(s))u_1(s)\Delta s, \int_{t_0}^{t} e_p(t,\sigma(s))u_1(s)\Delta s, 0 \right)^T,$$

where $u(\cdot) = (u_1(\cdot), u_2(\cdot), u_3(\cdot))^T \in L_1(\mathbb{T}_{t_0}, \mathbb{R}^3)$. Therefore,

$$\begin{aligned}
\|\mathbb{L}_{t_0}u\|_{L_1(\mathbb{T}_{t_0};\mathbb{R}^3)} &= 2\int_{t_0}^{\infty} \left| \int_{t_0}^{t} e_p(t,\sigma(s))u_1(s)\Delta s \right| \Delta t \\
&\le 2\int_{t_0}^{\infty} \left(\int_{t_0}^{t} \left| e_p(t,\sigma(s))u_1(s) \right| \Delta s \right) \Delta t \\
&= 2\int_{t_0}^{\infty} \left(\int_{\sigma(s)}^{\infty} e_p(t,\sigma(s))\Delta t \right) u_1(s)\Delta s \\
&= 2\int_{t_0}^{\infty} \frac{u_1(s)}{-p(\sigma(s))}\Delta s \le 8\int_{t_0}^{\infty} u_1(s)\Delta s \\
&\le 8\|u\|_{L_1(\mathbb{T}_{t_0},\mathbb{R}^3)}.
\end{aligned}$$

This implies that $\|\mathbb{L}_{t_0}\| \le 8$. Moreover, if we choose $u(\cdot) = (u_1(\cdot), 0, 0)^T$ with

$$u_1(t) = \begin{cases} 1 & \text{if } t = 3k, \\ 0 & \text{if } t \ne 3k, \end{cases}$$

for some k satisfying $3k > t_0$, then we get $\|\mathbb{L}_{t_0}u\|_{L_1(\mathbb{T}_{t_0},\mathbb{R}^3)} = 8\|u\|_{L_1(\mathbb{T}_{t_0},\mathbb{R}^3)}$. Therefore, $\|\mathbb{L}_{t_0}\| = 8$ for all t_0, and it follows that $\beta = \frac{1}{8}$. On the other hand, by (4.13),

$$\|\widetilde{\mathbb{L}}_{t_0}\| = \left\| \begin{bmatrix} 0 & -\frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & -1 \end{bmatrix} \right\| = 1, \text{ for all } t_0.$$

This implies that $\gamma = 1$. Thus, by Theorem 4.9, we obtain

$$r_{\mathbb{K}}(E_\sigma, A; B, C; \mathbb{T}) = \frac{1}{8}.$$

Let $\Sigma(\cdot)$ be a linear and causal operator from $L_p(\mathbb{T}_{t_0}, \mathbb{K}^q)$ to $L_p(\mathbb{T}_{t_0}, \mathbb{K}^m)$, i.e., $\Sigma(\cdot) \in L_\infty(\mathbb{T}_{t_0}, \mathbb{K}^{m\times q})$, defined by $(\Sigma u)(t) = \Sigma(t)u(t)$. Moreover, we have

$$\|\Sigma\| = \operatorname{esssup}_{t_0 \le t \le \infty} \|\Sigma(t)\|.$$

Therefore, we get a necessary condition for global L_p-stability of the equation (4.6) stated in the following corollary:

Corollary 4.13. *Let Assumptions* 4.1, 4.2 *hold true. If*

$$r_{\mathbb{K}}(E_\sigma, A; B, C; \mathbb{T}) > \operatorname{esssup}_{t_0 \le t \le \infty} \|\Sigma(t)\|,$$

then the perturbed equation (4.6) *is globally L_p-stable.*

Remark 4.14. In case $\mathbb{T} = \mathbb{R}$ and $E = I$ and $\Sigma(\cdot) \in L_\infty(\mathbb{R}_{t_0}, \mathbb{K}^{m\times q})$, the above corollary implies a lower bound for the stability radius in [39].

Remark 4.15. By the Fourier-Plancherel transformation technique as in [38, 58], if E, A, B, C are constant matrices and $p = 2$, then we can prove the equality

$$\|\mathbb{L}_{t_0}\| = \sup_{\lambda \in \partial S} \|C(A - \lambda E)^{-1} B\|,$$

where S is the domain of uniform exponential stability of the time scale $\mathbb{T}$,

$$S := \{\lambda \in \mathbb{C} : x^\Delta = \lambda x \text{ is uniformly exponentially stable}\}.$$

Moreover,

$$\|\widetilde{\mathbb{L}}_{t_0}\| = \lim_{\lambda \to \infty} \|C(A - \lambda E)^{-1} B\|.$$

Thus, in this case, we obtain the radius of stability formula in [28]

$$r(E, A; B, C; \mathbb{T}) = \frac{1}{\sup_{\lambda \in \partial S \cup \infty} \|C(A - \lambda E)^{-1} B\|}.$$

4.3 Stability Radius under Structured Perturbations on Both Sides

Now, in this section, we consider Equation (4.2) subject to perturbations acting on both the derivative and right-hand side of the form

$$E \rightsquigarrow \mathbb{E} := E + B_1\Sigma_1 C_1, \quad A \rightsquigarrow \mathbb{A} := A + B_2\Sigma_2 C_2,$$

where $B_i(\cdot) \in L_\infty(\mathbb{T}_{t_0}, \mathbb{K}^{n\times m}), C_i(\cdot) \in L_\infty(\mathbb{T}_{t_0}, \mathbb{K}^{q\times n})$ are given matrices, and $\Sigma_i(\cdot) \in L_\infty(\mathbb{T}_{t_0}, \mathbb{K}^{m\times q})$ are perturbations, for only $i = 1, 2$. Then the perturbed equation is

$$(E_\sigma + B_{1\sigma}\Sigma_{1\sigma}C_{1\sigma})(t)x^\Delta(t) = (A + B_2\Sigma_2 C_2)(t)x(t),\ t \geq t_0. \tag{4.17}$$

or

$$\mathbb{E}_\sigma(t)x^\Delta(t) = \mathbb{A}(t)x(t),\ t \geq t_0,$$

From the analysis in [7, 27], it is already known that for DAEs, it is necessary to restrict the structure of perturbation in order to get a meaningful problem of robust stability. Since under acting of arbitrary small perturbations, the solvability and/or the stability may be destroyed, due to the increasing of the system index. Therefore, we introduce and define the set of admissible perturbations

$$\mathbb{S} = \mathbb{S}(E; B_1, C_1) := \{(\Sigma_1, \Sigma_2) : \ker(E + B_1\Sigma_1 C_1) = \ker(E)\}.$$

We now prove the following lemmas.

Lemma 4.16. *The following assertions hold true.*

i) $Q_\sigma Q^\Delta H Q_\sigma = 0$;

ii) $Q_\sigma Q^\Delta P = Q^\Delta P$;

iii) $I + Q^\Delta H Q_\sigma$ *is invertible;*

iv) $(I + Q^\Delta H Q_\sigma)G^{-1} = (E_\sigma - AHQ_\sigma)^{-1}$, $Q_\sigma G^{-1} = Q_\sigma(E_\sigma - AHQ_\sigma)^{-1}$.

Proof. First we observe that

$$Q^\Delta = (QQ)^\Delta = Q^\Delta Q + Q_\sigma Q^\Delta. \tag{4.18}$$

Multiplying both sides of (4.18) by HQ_σ, we obtain

$$Q^\Delta HQ_\sigma = Q^\Delta QHQ_\sigma + Q_\sigma Q^\Delta HQ_\sigma.$$

Since $H|_{\ker E_\sigma}$ is a bounded isomorphism from $\ker E_\sigma$ to $\ker E$, and $QHQ_\sigma = HQ_\sigma$, we get

$$Q_\sigma Q^\Delta HQ_\sigma = 0.$$

To prove ii), it is clear that, by (4.18),

$$Q_\sigma Q^\Delta P = (Q^\Delta - Q^\Delta Q)P = Q^\Delta P - Q^\Delta QP = Q^\Delta P.$$

Next, to prove iii), we have, by i)

$$(Q^\Delta HQ_\sigma)^2 = Q^\Delta HQ_\sigma Q^\Delta HQ_\sigma = 0.$$

It implies that $(I + Q^\Delta HQ_\sigma)(I - Q^\Delta HQ_\sigma) = I$, and hence, $I + Q^\Delta HQ_\sigma$ is invertible.

Finally, to prove iv), remembering that $\bar{A} = A + E_\sigma P^\Delta = A - E_\sigma Q^\Delta$ yields

$$\begin{aligned}(E_\sigma - AHQ_\sigma)(I + Q^\Delta HQ_\sigma) &= E_\sigma - AHQ_\sigma + E_\sigma Q^\Delta HQ_\sigma - AHQ_\sigma Q^\Delta HQ_\sigma \\ &= E_\sigma - AHQ_\sigma + E_\sigma Q^\Delta HQ_\sigma \\ &= E_\sigma - \bar{A}HQ_\sigma = G.\end{aligned}$$

Therefore, $E_\sigma - AHQ_\sigma$ is invertible and

$$(I + Q^\Delta HQ_\sigma)G^{-1} = (E_\sigma - AHQ_\sigma)^{-1}.$$

Moreover, we have

$$Q_\sigma G^{-1} = Q_\sigma(I + Q^\Delta HQ_\sigma)G^{-1} = Q_\sigma(E_\sigma - AHQ_\sigma)^{-1}.$$

The proof is complete. □

To be continue, we define $\bar{\mathbb{A}} = \mathbb{A} - \mathbb{E}_\sigma Q^\Delta, \quad \mathbb{G} := \mathbb{E}_\sigma - \bar{\mathbb{A}}HQ_\sigma$ and

$$B := \begin{bmatrix} B_{1\sigma} & B_2 \end{bmatrix}, \quad F := \begin{bmatrix} C_{1\sigma}P_\sigma(E_\sigma - AHQ_\sigma)^{-1} \\ -C_2HQ_\sigma(E_\sigma - AHQ_\sigma)^{-1} \end{bmatrix},$$
$$\Sigma_b := \begin{bmatrix} \Sigma_{1\sigma} & 0 \\ 0 & \Sigma_2 \end{bmatrix}, \quad \bar{C} := \begin{bmatrix} C_{1\sigma}P_\sigma Q^\Delta \\ -C_2 \end{bmatrix}, \quad C := (F\bar{A} + \bar{C})P.$$

Lemma 4.17. *Assume that Equation* (4.2) *is of* index-1. *If* $(\Sigma_1, \Sigma_2) \in \mathbb{S}$ *such that* $\|\Sigma_b\| < \dfrac{1}{\|FB\|}$, *then the perturbed equation* (4.17) *is also of* index-1.

Proof. We have

$$B_1\Sigma_1 C_1 = B_1\Sigma_1 C_1 P, \ B_{1\sigma}\Sigma_{1\sigma}C_{1\sigma} = B_{1\sigma}\Sigma_{1\sigma}C_{1\sigma}P_\sigma,$$

for every $(\Sigma_1, \Sigma_2) \in \mathbb{S}$. Furthermore,

$$\begin{aligned}\bar{\mathbb{A}} = \mathbb{A} - \mathbb{E}_\sigma Q^\Delta &= A - E_\sigma Q^\Delta + B_2\Sigma_2 C_2 - B_{1\sigma}\Sigma_{1\sigma}C_{1\sigma}Q^\Delta \\ &= \bar{A} + B_2\Sigma_2 C_2 - B_{1\sigma}\Sigma_{1\sigma}C_{1\sigma}Q^\Delta \\ &= \bar{A} + B_2\Sigma_2 C_2 - B_{1\sigma}\Sigma_{1\sigma}C_{1\sigma}P_\sigma Q^\Delta \\ &= \bar{A} - B\Sigma_b\bar{C}.\end{aligned}$$

Therefore, we get

$$\begin{aligned}\mathbb{G} = \mathbb{E}_\sigma - \bar{\mathbb{A}}HQ_\sigma &= E_\sigma + B_{1\sigma}\Sigma_{1\sigma}C_{1\sigma} - (\bar{A} + B_2\Sigma_2 C_2 - B_{1\sigma}\Sigma_{1\sigma}C_{1\sigma}Q^\Delta)HQ_\sigma \\ &= G - B_2\Sigma_2 C_2 HQ_\sigma + B_{1\sigma}\Sigma_{1\sigma}C_{1\sigma}P_\sigma(I + Q^\Delta HQ_\sigma) \\ &= G + B\Sigma_b \begin{bmatrix} C_{1\sigma}P_\sigma(I + Q^\Delta HQ_\sigma) \\ -C_2 HQ_\sigma \end{bmatrix} \\ &= \left(I + B\Sigma_b \begin{bmatrix} C_{1\sigma}P_\sigma(I + Q^\Delta HQ_\sigma)G^{-1} \\ -C_2 HQ_\sigma G^{-1} \end{bmatrix} \right) G \\ &= \left(I + B\Sigma_b \begin{bmatrix} C_{1\sigma}P_\sigma(E_\sigma - AHQ_\sigma)^{-1} \\ -C_2 HQ_\sigma(E_\sigma - AHQ_\sigma)^{-1} \end{bmatrix} \right) G \\ &= (I + B\Sigma_b F)G.\end{aligned}$$

On the other hand, if $\|\Sigma_b\| < \frac{1}{\|FB\|}$ then $I + \Sigma_b FB$ is invertible. By Lemma 4.2, $I + B\Sigma_b F$ is invertible. Therefore, so is $\mathbb{G}$. Thus, the perturbed equation (4.17) is also of index-1. The proof is complete. □

Lemma 4.18. *Let Equation* (4.2) *be of* index-1. *Then Equation* (4.6) *is equivalent to Equation* (4.17) *with the perturbation* $\Sigma = (I + \Sigma_b FB)^{-1}\Sigma_b$.

Proof. Due to the Lemma 4.2 and the proof of Lemma 4.17, we have

$$\mathbb{G}^{-1} = G^{-1}[I - B(I + \Sigma_b FB)^{-1}\Sigma_b F] = G^{-1} - G^{-1}B(I + \Sigma_b FB)^{-1}\Sigma_b F.$$

Note that

$$\begin{aligned}[I - B(I+\Sigma_b FB)^{-1}\Sigma_b F]B &= B[I - (I+\Sigma_b FB)^{-1}\Sigma_b FB] \\ &= B[I - (I+\Sigma_b FB)^{-1}(\Sigma_b FB + I) + (I+\Sigma_b FB)^{-1}] \\ &= B(I+\Sigma_b FB)^{-1}.\end{aligned}$$

Therefore,

$$\begin{aligned}\mathbb{G}^{-1}\bar{\mathbb{A}} &= (G^{-1} - G^{-1}B(I+\Sigma_b FB)^{-1}\Sigma_b F)(\bar{A} - B\Sigma_b\bar{C}) \\ &= G^{-1}\bar{A} - G^{-1}B(I+\Sigma_b FB)^{-1}\Sigma_b F\bar{A} \\ &\qquad - G^{-1}(I - B(I+\Sigma_b FB)^{-1}\Sigma_b F)B\Sigma_b\bar{C} \\ &= G^{-1}\bar{A} - G^{-1}B(I+\Sigma_b FB)^{-1}\Sigma_b F\bar{A} - G^{-1}B(I+\Sigma_b FB)^{-1}\Sigma_b\bar{C} \\ &= G^{-1}\bar{A} - G^{-1}B(I+\Sigma_b FB)^{-1}\Sigma_b(F\bar{A}+\bar{C}).\end{aligned}$$

This implies that

$$\begin{aligned}\mathbb{G}^{-1}\bar{\mathbb{A}}P &= G^{-1}\bar{A}P - G^{-1}B(I+\Sigma_b FB)^{-1}\Sigma_b(F\bar{A}+\bar{C})P \\ &= G^{-1}\bar{A}P + G^{-1}B\Sigma C,\end{aligned}$$

where $\Sigma = (I+\Sigma_b FB)^{-1}\Sigma_b, C = (F\bar{A}+\bar{C})P$. Similar to the decomposition into (3.3), (3.4) (see Section 3.1, Chapter 3) with $f = 0$, we see that the perturbed equation (4.17) is equivalent to the system

$$\begin{cases}(Px)^\Delta &= (P^\Delta + P_\sigma\mathbb{G}^{-1}\bar{\mathbb{A}})Px, \\ Qx &= HQ_\sigma\mathbb{G}^{-1}\bar{\mathbb{A}}Px.\end{cases} \tag{4.19}$$

Replace $\mathbb{G}^{-1}\bar{\mathbb{A}}P$ by $G^{-1}\bar{A}P + G^{-1}B\Sigma C$ in the system (4.19), we get

$$\begin{cases}(Px)^\Delta &= (P^\Delta + P_\sigma G^{-1}\bar{A})Px + P_\sigma G^{-1}B\Sigma Cx, \\ Qx &= HQ_\sigma G^{-1}\bar{A}Px + HQ_\sigma G^{-1}B\Sigma Cx.\end{cases} \tag{4.20}$$

By the analysis of implicit dynamic equation in Chapter 3, it is clear that the system (4.20) is equivalent to Equation (4.6). The proof is complete. □

Definition 4.19. Let Assumptions 4.1, 4.2 hold. The complex (real) structured stability radius of Equation (4.2) subject to linear structured perturbations in Equation (4.17) is defined by

$$r_{\mathbb{K}}(E_\sigma, A; B_1, C_1, B_2, C_2; \mathbb{T}) = \inf\left\{\begin{array}{c}\|\Sigma_b\|, \text{ the trivial solution of (4.17) is not} \\ \text{globally } L_p\text{-stable or (4.17) is not of index-1}\end{array}\right\}.$$

Theorem 4.20. *Let Assumptions* 4.1, 4.2 *hold, and* β, γ *be defined in* (4.13). *The complex* (*real*) *structured stability radius of Equation* (4.2) *subject to linear structured perturbations in Equation* (4.17) *satisfies*

$$r_{\mathbb{K}}(E_\sigma, A; B_1, C_1, B_2, C_2; \mathbb{T}) \geq \begin{cases} \dfrac{\min\{\beta;\gamma\}}{1+\|FB\|\min\{\beta;\gamma\}} & \text{if } \beta < \infty \text{ or } \gamma < \infty, \\ \dfrac{1}{\|FB\|} & \text{if } \beta = \infty \text{ and } \gamma = \infty. \end{cases}$$

Proof. Firstly, consider the case that either $\beta < \infty$ or $\gamma < \infty$. Assume that

$$\|\Sigma_b\| < \frac{\min\{\beta;\gamma\}}{1+\|FB\|\min\{\beta;\gamma\}}.$$

This implies that $\|\Sigma_b\| < \dfrac{1}{\|FB\|}$. By Lemma 4.17, we see that the perturbed equation (4.17) is of index-1. With $\Sigma = (I + \Sigma_b FB)^{-1}\Sigma_b$, we have

$$\|\Sigma\| \leq \frac{\|\Sigma_b\|}{1 - \|\Sigma_b\|\|FB\|} < \min\{\beta;\gamma\}.$$

Therefore, by Corollary 4.13, the perturbed equation (4.6) is globally L_p-stable. Thus, by Lemma 4.18, Equation (4.17) is also globally L_p-stable. This implies that

$$r_{\mathbb{K}}(E_\sigma, A; B_1, C_1, B_2, C_2; \mathbb{T}) \geq \frac{\min\{\beta;\gamma\}}{1+\|FB\|\min\{\beta;\gamma\}}.$$

Finally, suppose that $\beta = \infty$ and $\gamma = \infty$. If $\|\Sigma_b\| < \dfrac{1}{\|FB\|}$ then by Lemma 4.17, it follows that the perturbed equation (4.17) has index-1 and is also globally L_p-stable. Thus, we also get

$$r_{\mathbb{K}}(E_\sigma, A; B_1, C_1, B_2, C_2; \mathbb{T}) \geq \frac{1}{\|FB\|}.$$

The proof is complete. □

Example 4.21. Consider the implicit dynamic equation $E_\sigma x^\Delta = Ax$, with

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, A = \begin{bmatrix} -1 & \frac{1}{2} & 0 \\ \frac{1}{2} & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix}.$$

Assume that this equation is subject to structured perturbations as follows

$$E \rightsquigarrow \mathbb{E} = \begin{bmatrix} 1+\delta_1(t) & \delta_1(t) & \delta_1(t) \\ \delta_1(t) & 1+\delta_1(t) & \delta_1(t) \\ 0 & 0 & 0 \end{bmatrix},$$

$$A \rightsquigarrow \mathbb{A} = \begin{bmatrix} -1 & \frac{1}{2} & 0 \\ \frac{1}{2}+\delta_2(t) & -1+\delta_2(t) & 1+\delta_2(t) \\ \delta_2(t) & \delta_2(t) & -1+\delta_2(t) \end{bmatrix},$$

where $\delta_i(t), i = 1, 2$, are perturbations. We can directly see that this model can be rewritten in form (4.17) with

$$B_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, B_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, C_1 = C_2 = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}.$$

In this example, we choose

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, Q = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

By simple computations, we get

$$B = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}, F = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, C = \begin{bmatrix} -\frac{1}{2} & -\frac{1}{2} & 0 \\ -1 & -1 & 0 \end{bmatrix}.$$

Therefore

$$\|FB\| = \left\| \begin{bmatrix} 2 & 1 \\ 0 & -1 \end{bmatrix} \right\|_\infty = 3$$

and

$$C(A-\lambda E)^{-1}B = \frac{1}{(\lambda+1)^2 - \frac{1}{4}} \begin{bmatrix} \lambda+\frac{3}{2} & \lambda+\frac{3}{2} \\ 2\lambda+3 & 2\lambda+3 \end{bmatrix}.$$

Let $\mathbb{T} = \bigcup_{k=1}^{\infty}[2k, 2k+1]$. Then, the domain of uniformly exponential stability

$$S = \{\lambda \in \mathbb{C} : \Re\lambda + \ln|1+\lambda| < 1\} \text{ (see [28])}.$$

Using Remark 4.15, we yield

$$\beta = \|\mathbb{L}_\infty\|^{-1} = \frac{1}{\sup_{\lambda \in \partial S} \|C(A-\lambda E)^{-1}B\|_\infty} = \frac{1}{\|CA^{-1}B\|_\infty} = \frac{1}{8},$$

$$\gamma = \|\widetilde{\mathbb{L}}_a\|^{-1} = \frac{1}{\lim_{\lambda\to\infty} \|C(A-\lambda E)^{-1}B\|_\infty} = +\infty.$$

Thus, by applying Theorem 4.20, we obtain $r_{\mathbb{K}}(E_\sigma, A; B_1, C_1, B_2, C_2; \mathbb{T}) \geq \dfrac{1}{11}$.

In the rest of this section, we assume that the perturbed equation (4.17) is given by unstructured perturbations with $B_1 = B_2 = C_1 = C_2 = I$. Let

$$\begin{aligned} l(E,A) &:= 2 \lim_{t\to\infty} \|\mathbb{M}_t\|^{-1}, \\ k_1 &:= \left\| \begin{bmatrix} -P_\sigma(E_\sigma - AHQ_\sigma)^{-1}A(P - HQ^\Delta P) \\ (I + HQ_\sigma(E_\sigma - AHQ_\sigma)^{-1}A)(P - HQ^\Delta P) \end{bmatrix} \right\|_\infty, \\ k_2 &:= \left\| \begin{bmatrix} P_\sigma(E_\sigma - AHQ_\sigma)^{-1} \\ -HQ_\sigma(E_\sigma - AHQ_\sigma)^{-1} \end{bmatrix} \right\|_\infty. \end{aligned}$$

Corollary 4.22. *Let Assumptions* 4.1, 4.2 *hold. Then, the complex (real) structured stability radius of Equation* (4.2) *subject to linear unstructured perturbations* $E \rightsquigarrow E + \Sigma_1,\ A \rightsquigarrow A + \Sigma_2$ *satisfies*

$$r_{\mathbb{K}}(E_\sigma, A; I; \mathbb{T}) \geq \begin{cases} \dfrac{\min\{l(E,A), \|HQ_\sigma G^{-1}\|_\infty^{-1}\}}{k_1 + k_2 \min\{l(E,A), \|HQ_\sigma G^{-1}\|_\infty^{-1}\}} & \text{if } Q \neq 0 \text{ or } l(E,A) < \infty, \\ \dfrac{1}{k_2} & \text{if } Q = 0 \text{ and } l(E,A) = \infty. \end{cases}$$

with the convention $\|HQ_\sigma G^{-1}\|_\infty^{-1} = \infty$ *if* $\|HQ_\sigma G^{-1}\|_\infty = 0$.

Proof. Since $B_1 = B_2 = C_1 = C_2 = I$, we have

$$\begin{aligned} \|FB\| &= \left\| \begin{bmatrix} P_\sigma(E_\sigma - AHQ_\sigma)^{-1} \\ -HQ_\sigma(E_\sigma - AHQ_\sigma)^{-1} \end{bmatrix} \begin{bmatrix} I & I \end{bmatrix} \right\|_\infty \\ &= \left\| \begin{bmatrix} P_\sigma(E_\sigma - AHQ_\sigma)^{-1} & P_\sigma(E_\sigma - AHQ_\sigma)^{-1} \\ -HQ_\sigma(E_\sigma - AHQ_\sigma)^{-1} & -HQ_\sigma(E_\sigma - AHQ_\sigma)^{-1} \end{bmatrix} \right\|_\infty \\ &= \left\| \begin{bmatrix} P_\sigma(E_\sigma - AHQ_\sigma)^{-1} \\ -HQ_\sigma(E_\sigma - AHQ_\sigma)^{-1} \end{bmatrix} \right\| = 2k_2, \end{aligned}$$

$$\begin{aligned} \|C\| &= \|(F\bar{A} + \bar{C})P\|_\infty \\ &= \left\| \begin{bmatrix} P_\sigma(E_\sigma - AHQ_\sigma)^{-1} \\ -HQ_\sigma(E_\sigma - AHQ_\sigma)^{-1}) \end{bmatrix} (A - E_\sigma Q^\Delta)P + \begin{bmatrix} P_\sigma Q^\Delta P \\ -P \end{bmatrix} \right\|_\infty. \end{aligned}$$

Using the equality ii) in Lemma 4.16, we have

$$\begin{aligned}(A - E_\sigma Q^\Delta)P &= A(P - HQ_\sigma Q^\Delta P) - (E_\sigma - AHQ_\sigma)Q^\Delta P \\ &= A(P - HQ^\Delta P) - (E_\sigma - AHQ_\sigma)Q^\Delta P.\end{aligned}$$

This implies that

$$\|C\| = \left\| - \begin{bmatrix} -P_\sigma(E_\sigma - AHQ_\sigma)^{-1}A(P - HQ^\Delta P) \\ (I + HQ_\sigma(E_\sigma - AHQ_\sigma)^{-1}A)(P - HQ^\Delta P) \end{bmatrix} \right\|_\infty = k_1,$$

and

$$\|\mathbb{L}_t\| = \|C(\cdot)\mathbb{M}_t\| \le k_1\|\mathbb{M}_t\|.$$

Therefore,

$$\beta \ge \frac{l(E, A)}{2k_1}.$$

Moreover,

$$\begin{aligned}\gamma = \|\widetilde{\mathbb{L}}_a\|^{-1} = \|CHQ_\sigma G^{-1}B\|_\infty^{-1} &\ge \|C\|_\infty^{-1}\|B\|_\infty^{-1}\|HQ_\sigma G^{-1}\|_\infty^{-1} \\ &\ge \frac{1}{2k_1}\|HQ_\sigma G^{-1}\|_\infty^{-1}.\end{aligned}$$

On the other hand, if $Q = 0$ then $\|\widetilde{\mathbb{L}}_a\| = 0$, and if $l(E, A) = \infty$ then $\|\mathbb{L}_\infty\| = \beta^{-1} = 0$.

Consequently, Corollary 4.22 follows from Theorem 4.20. The proof is complete. □

Remark 4.23. In case $\mathbb{T} = \mathbb{R}$, this corollary is a result concerning the lower bound of the stability radius in [7, Theorem 6.11].

Example 4.24. Consider Equation (4.2) with $E, A, \mathbb{T}$ in Example 4.12. Then, we can compute

$$P_\sigma(E_\sigma - AHQ_\sigma)^{-1}A(P - HQ^\Delta P) = \begin{bmatrix} \frac{p}{2} & \frac{p}{2} & 0 \\ \frac{p}{2} & \frac{p}{2} & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$(I + HQ_\sigma(E_\sigma - AHQ_\sigma)^{-1}A)(P - HQ^\Delta P) = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$P_\sigma(E_\sigma - AHQ_\sigma)^{-1} = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$
$$HQ_\sigma(E_\sigma - AHQ_\sigma)^{-1} = \begin{bmatrix} 0 & -\frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

Since $\|p\|_\infty = \frac{1}{2}$, it is easy to imply that $k_1 = k_2 = 1$. Hence, by Corollary 4.22, we obtain

$$r_{\mathbb{K}}(E_\sigma, A; I; \mathbb{T}) \geq \frac{\frac{1}{8}}{1 + \frac{1}{8}} = \frac{1}{9}.$$

Conclusions of Chapter 4. In this chapter, we have investigated the robust stability for linear time-varying implicit dynamic equations on time scales. The main results of Chapter 4 are:

1. Establishing the structured stability radius formula of the IDEs with respect to dynamic perturbations in Theorem 4.9, and a lower bounded in Corollary 4.13;
2. Recommending the lower bounds for the stability radius involving structured perturbations acting on both sides in Theorem 4.20, and Corollary 4.22.
3. Extending previous results for the stability radius of time-varying differential, difference equations, differential-algebraic and implicit difference equations for general time scales in Remarks 4.10, 4.11, 4.14, and 4.15.

The results got in this chapter are the extensions of many previous ones for the stability radius of linear systems. We will continue to study the stabilization and other control properties in a control frame for linear time-varying implicit dynamic equations in the next time.

CONCLUSIONS

1. Achieved results: The thesis studies the stability and robust stability of linear time-varying implicit dynamical equations. The following results have achieved:

1. Introducing of the definition for Lyapunov exponent and using it to study the stability of linear dynamic equations on time scales.
2. Establishing the robust stability of implicit dynamic equations with Lipschitz perturbations, and extending Bohl-Perron type stability theorem for implicit dynamic equations on time scales.
3. Suggesting the concept for Bohl exponent on time scales and studying the relation between exponential stability and the Bohl exponent when dynamic equations under perturbations acting on the system coefficients.
4. Recommending the radius of stability formula for implicit dynamic equations on time scales under some structured perturbations acting on the right-hand side or both side-hands.

2. Outlooks: In the future, results in this dissertation could be extended in some following directions:

1. Using the Lyapunov exponent to investigate the stability of non-linear dynamical systems.
2. Investigating the relation between Bohl exponent and robust stability for implicit dynamic equations under non-linear perturbations.
3. Studying the stabilization and other control properties in a control frame for implicit dynamic equations.

LIST OF THE AUTHOR'S SCIENTIFIC WORKS

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