MINISTRY OF EDUCATION AND TRAINING
HANOI PEDAGOGICAL UNIVERSITY 2

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SOME METHODS FOR SOLVING VARIATIONAL INEQUALITY PROBLEMS OVER FIXED POINT SETS

SUMMARY OF DOCTORAL THESIS

Ha Noi - 2024

This thesis has been completed at Hanoi Pedagogical University 2, based on the research results of the author and colleagues.

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Reviewer 3:...

The thesis will be defended at the Board of Examiners of Hanoi Pedagogical University 2
at : : : on : : : , : : : of : : : , : : : .

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Introduction

1. Overview of research situation

In 1966, while studying boundary problems for a class of partial differential equations, Hartman, Ph. and Stampacchia, G. first mentioned the variational inequality model. Then, this problem became known for its interesting applications such as Nash economic equilibrium model, traffic network equilibrium model, optimal routing model of communication network, non-cooperative game theory, image processing model and many other applications described by Kinderlehrer, D. and Stampacchia, G. in the popular book "An Introduction to Variational Inequalities and Their Application" and in some other documents.Variational inequality problems contain many familiar classes of problems, such as subdifferential convex optimization problems, Kakutani fixed point problems, nonlinear compensation problems, and several other models.

Let C be a nonempty closed convex subset of a real Hilbert space $\mathbb{H}$ and a map $F : \mathbb{H} \to \mathbb{H}$ (often called the cost mapping), the variational inequality problem with the cost mapping F and the constraint domain C, denoted $VI(C, F)$, is stated as:

$$\text{Find } x^* \in C \text{ such that } \langle F(x^*), x - x^* \rangle \geq 0, \quad \forall x \in C.$$

The variational inequality problem $VI(C, F)$ is a popular research subject in the fields of Analysis and Optimization Theory. Currently, there are two main research directions on this problem. *First is*, qualitative research on the existence of solutions and properties of the solution set of the problem. Outstanding results of this research direction must be mentioned with domestic and foreign research groups of Professor Yen, N.D. et.al, Professor Khanh, P.Q. et.al, Professor Mordukhovich, B.S. et.al and many other authors. *Second is*, research proposes solving algorithms and direct applications to specific models. One of the popular methods to solve this problem is the one-projection method, with an iteration diagram of the form:

$$\begin{cases} x^0 \in C, \\ x^{k+1} = \Pi_C[x^k - \lambda F(x^k)], \quad \forall k \geq 0. \end{cases}$$

Under the assumption that the cost mapping F is $\beta-$strongly monotone and $L-$Lipschitz-continuous, $\lambda \in (0, \frac{2\beta}{L^2})$, the sequence $\{x^k\}$ strongly converges to a unique solution x^* of problem $VI(C, F)$.

Let C be a convex, closed, and nonempty subset of a real Hilbert space $\mathbb{H}, I = \{1, 2, \cdots\}$, the maps $S_i : \mathbb{H} \to \mathbb{H}$, $(i \in I)$. In this thesis, we study and propose new algorithms to solve the variational inequality problem over fixed points set, denoted as $VIF(\Omega, F)$, which is stated as follows:

$$\text{Find } x^* \in \Omega \text{ such that } \langle F(x^*), x - x^* \rangle \geq 0, \quad \forall x \in \Omega, \tag{1}$$

where $Fix(S_i) := \{x \in \mathbb{H} : x = S_i(x)\}$ and $\Omega = \cap_{i \in I} Fix(S_i)$. Clearly, when S_i is an identity map, the problem $VIF(\Omega, F)$ is written as the usual variational inequality problem $VI(C, F)$.

When Ω is the fixed point set of a non-expansive map $S : \mathbb{H} \to \mathbb{H}$, Yamada, I. proposed the following quite simple gradient descent algorithm with the following iteration sequence:

$$\begin{cases} x^0 \in \mathbb{H}, \\ x^{k+1} = S(x^k) - \lambda F(S(x^k)), \quad \forall k \geq 0. \end{cases}$$

Under the assumptions of $\beta-$strongly monotone and $L-$Lipschitz continuous of the cost mapping F, and $\lambda \in (0, \frac{2\beta}{L^2})$, the iteration sequence $\{x^k\}$ strongly converges to a unique solution of the problem $VIF(\Omega, F)$. Extending this result of Yamada, I., Xu, H.K. proposed an iterative rotation algorithm with the constraint domain of the problem as a fixed point set of a finite family of non-expansive maps. Subsequently, Iemoto, S. and Takahashi, W. studied an extension of the algorithm for a constraint domain that is an infinite family of non-expansive maps. Some interesting research results on general monotone cost mappings and the constraint domain Ω is the intersection of pseudo-contraction maps (or quasi-nonexpansive maps) proposed by many authors, such as approximation methods, inertial relaxation methods, projection methods under derivatives, inertial contraction methods and some others.

As we know, a point $x^* \in C$ is a solution of the variational inequality problem $VI(C,G)$ (here $G : \mathbb{H} \to \mathbb{H}$) if and only if it is a fixed point of the solution mapping $S : \mathbb{H} \to C$ as follows:

$$S(x) = \Pi_C[x - \lambda G(x)], \quad \forall x \in \mathbb{H},$$

where $\lambda > 0$. In this case, the variational inequality problem on the fixed point set $VIF(\Omega, F)$ with $\Omega := Fix(S)$, denoted by $BVI(C,F,G)$, as follows:

$$\text{Find } u \in \Omega \text{ such that } \langle F(u), x - u\rangle \geq 0, \quad \forall x \in \Omega,$$

where $\Omega = \{x^* \in C : \langle G(x^*), y - x^*\rangle \geq 0, \quad \forall y \in C\}$.

The problem $VIF(\Omega, F)$ is a difficult problem, because the constraint domain Ω is the intersection of fixed point sets of the maps and is not given in explicit form. According to our understanding of current algorithms for solving variational inequality problems on fixed point set $VIF(\Omega, F)$, there are some salient points as follows:

- Convergence of the algorithms requires strongly monotone and Lipschitz continuous assumptions of the cost mapping F or some (rather complicated) regular monotonicity assumption;

- Iterative algorithms are not really efficient on computers, when the constraint domain C is complex. At each iteration in some algorithms, the iteration sequence is computed as a solution to another variational inequality problem. The convergence of the algorithm requires the exact solution at each iteration, however, the actual computation on the computer can only give an approximate solution with errors;

- Algorithms and calculations applied to practical models still have many limitations.

For the above reasons, we chose to research the thesis topic *"Some methods for solving variational inequality problems over fixed point sets"*. With the main goal of proposing new algorithms to solve the $VIF(\Omega, F)$ problem, we have studied and expanded the methods in optimization and fixed point iteration techniques, such as hybrid techniques, inertial techniques, iteration techniques, relaxed projection techniques, ... The strong and weak convergence of the proposed algorithms is demonstrated, illustrative examples and comparisons with other popular results are programmed and calculated on Matlab software.

The content of the thesis is written based on the results of 4 articles, of which 3 articles were published in SCIE journals ranked Q2, 1 article was published in Scopus journal.

In addition to the table of contents, list of symbols and abbreviations, introduction and references, the main content of the thesis is divided into 3 chapters as follows:

- Chapter 1: Variational inequality problems on fixed point sets.
- Chapter 2: Inertial techniques
- Chapter 3: Relaxed solution mapping method.

Chapter 1

Variational inequality problem on fixed point set

In the first chapter of the thesis, we review some basic concepts and knowledge in functional analysis and convex analysis, as a basis for research in the following chapters. In addition, the concepts of variational inequality problems over fixed point sets, the existence of solutions, auxiliary lemmas, image processing models and some common methods to solve variational inequality problems over fixed point sets such as: Iteration method, hybrid method, projection method are also presented in this chapter. The content of chapter 1 is written based on some results in reference documents Pham Ngoc Anh (2015) Hoang Tuy (2003), Bauschke, H.H., Combettes, P.L. (2011), Carl, S., Le, V.K. (2021), Ding, X.P., Lin, Y.C., Yao, J.C. (2007),...

1.1 Some basic concept

1.1.1 Projection and monotone mapping

1.1.2 Variational inequality problem

1.1.3 Fixed point problem

1.1.4 Some basic lemmas

1.2 Variational inequality problem on fixed point set

1.2.1 Problem statement and examples

1.2.2 Some special cases

1.2.3 Image processing model

1.2.4 Some common algorithms to solve the problem $VIF(\Omega, F)$

Chapter 2

Inertial techniques

2.1 Hybrid inertial contraction technique

In an extended study of the variational inequality problem $VI(C, F)$, Yamada, I. replaced the constraint domain C with the fixed point set of a mapping, the solution of the problem was found through the hybrid steepest descent algorithm *(HSDA)*. From this method, we proposed the algorithm *(HICA)* to solve the variational inequality problem over fixed point set $VIF(\Omega, F)$, the algorithm is a combination of the inertial technique and the hybrid method. The strong convergence result is proven in a real Hilbert space $\mathbb{H}$. The $(PIPA)$ algorithm is a combination of parallel computing techniques with inertial techniques to solve the variational inequality problem over fixed point set $VIF(\Omega, F)$, where Ω is the intersection of the fixed point sets of the demicontractive maps $\Omega = \bigcap_{i\in J} FixS_i$, $J = \{1, 2, \ldots, m\}$. The last part shows illustrative calculations, applications in image processing models and comparisons with some popular algorithms.

The content of this chapter is written based on two articles $[CT1]$ and $[CT3]$ (List of works related to the thesis).

2.1.1 Algorithm *(HICA)*

Let $F : \mathbb{H} \to \mathbb{H}$ be a cost mapping and let the family of mappings $S_i : \mathbb{H} \to \mathbb{H}$, $\forall i \in I$ satisfy the following assumptions:

(A_1) The mapping F is β-strongly monotone and L-Lipschitz-continuous;

(A_2) For every $i \in I$, S_i is ξ_i-demicontractive, satisfying condition (Z) and

$$\Omega := \bigcap_{i \in I} \mathrm{Fix}(S_i) \neq \emptyset;$$

(A_3) For every $k \geq 0$, the positive parameters β_k, γ_k, τ_k, λ_k and μ_k satisfy

$$\begin{cases} 0 < c_1 \leq \beta_k \leq c_2 < 1, \ \mu_k \leq \eta, \\ \alpha_k \in (0, 1 - \xi_k], \ \inf_k \alpha_k > 0, \\ 0 < \gamma_k < 1, \ \lim\limits_{k \to \infty} \gamma_k = 0, \ \sum_{k=1}^{\infty} \gamma_k = \infty, \\ \lim\limits_{k \to \infty} \frac{\tau_k}{\gamma_k} = 0, \ \lambda_k \in \left(\frac{\beta}{L^2}, \frac{2\beta}{L^2} \right), \ a \in (0,1), \\ \sqrt{1 - 2\lambda_k \beta + \lambda_k^2 L^2} < 1 - a. \end{cases} \tag{2.1}$$

Algorithm 2.1. *Hybrid inertial contraction algorithm (HICA)*
Initialization: Let x^0, $x^1 \in \mathbb{H}$ be arbitrary. At the kiteration, $k = 1, 2, \ldots$.
Step 1. Compute the inertial coefficient

$$\theta_k = \begin{cases} \min \left\{ \mu_k, \dfrac{\tau_k}{\|x^k - x^{k-1}\|} \right\} & \textit{if } \|x^k - x^{k-1}\| \neq 0, \\ \mu_k & \textit{otherwise,} \end{cases} \tag{2.2}$$

Step 2. Compute

$$\begin{cases} w^k = x^k + \theta_k (x^k - x^{k-1}), \\ \bar{S}_k w^k = (1 - \alpha_k) w^k + \alpha_k S_k w^k, \\ z^k = (1 - \gamma_k) \bar{S}_k w^k + \gamma_k \left[w^k - \lambda_k F(w^k) \right], \\ \bar{S}_k z^k = (1 - \alpha_k) z^k + \alpha_k S_k z^k, \\ x^{k+1} = (1 - \beta_k) \bar{S}_k w^k + \beta_k \bar{S}_k z^k. \end{cases} \tag{2.3}$$

Step 3. Set $k := k + 1$ and go back to Step 1.

2.1.2 Convergence theorem

Theorem 2.1. *Suppose that the assumptions $(A_1), (A_2)$ and (A_3) are satisfied. Then the iteration sequence $\{x^k\}$ given by Algorithm 2.1 strongly converges to the unique solution x^* of the problem $VIF(\Omega, F)$.*

2.1.3 Calculation examples

In this section, we present some numerical experiments on computers to illustrate the convergence of the $(HICA)$ algorithm and compare it with two algorithms: the Parallel Projection Algorithm (PPA) by Anh, P.N. and colleagues and the Hybrid Gradient Algorithm $(HSDA)$ by Yamada, I. All computational experiments are programmed in MATLAB R2016a, running on a PC with an Intel® Core™ i7-7800X CPU @ 3.50 GHz 32 GB Ram.

2.2 Inertial parallel approximation technique

2.2.1 Algorithm *(PIPA)*

Given $x^0, x^1 \in \mathbb{H}$ and the parameters satisfy the following conditions

$$\begin{cases} a \in (0,1), \{\lambda_k\} \subset [\overline{a}, \hat{a}] \subset \left(0, \dfrac{2\beta}{L^2}\right), \\ \sqrt{1 - 2\lambda_k \beta + \lambda_k^2 L^2} < 1 - a, \\ \zeta_k \in (0,1), \sum\limits_{k=1}^{\infty} \zeta_k = \infty, \lim\limits_{k\to\infty} \zeta_k = 0, \\ 0 \le \tau_k \le \zeta_k^2, \mu_k > 0, \\ \gamma_{k,i} \in (\overline{b}, \hat{b}) \subset (0, 1 - \max\{\beta_i : i \in J\}). \end{cases} \tag{2.4}$$

Step 1: Compute:

$$w^k = x^k + \alpha_k (x^k - x^{k-1}), \tag{2.5}$$

with

$$\alpha_k = \begin{cases} \min\left\{ \dfrac{\tau_k}{\|x^k - x^{k-1}\|}, \mu_k \right\} & \text{if } \|x^k - x^{k-1}\| \neq 0, \\ \mu_k & \text{otherwise.} \end{cases} \tag{2.6}$$

Step 2: Compute $u_i^k = (1 - \gamma_{k,i})\, w^k + \gamma_{k,i} S_i w^k$.
Set $t^k = u_{i_0}^k$, where $i_0 \in \arg\max \left\{ \left\| u_i^k - w^k \right\| : i \in J \right\}$.
Step 3: Compute

$$x^{k+1} = (1 - \varsigma_k)\, t^k + \varsigma_k \left[t^k - \lambda_k F\left(t^k\right) \right]. \text{ Set } k = k+1 \text{ and go to Step 1.}$$

2.2.2 Convergence theorem

Theorem 2.2. *Let $F : \mathbb{H} \to \mathbb{H}$ be a β_i-strongly monotone and L-Lipschitz continuous map and the family $S_i : \mathbb{H} \to \mathbb{H}$ be demicontractive and demi-closed at 0 maps for all $i \in J$. Under the conditions (2.4) and $\Omega \neq \emptyset$, the sequence $\{x^k\}$ given by $(PIPA)$ Algorithm strongly converges to the unique solution x^* of the problem $VIF(\Omega, F)$.*

2.2.3 Apply to image restoration model

In this section, we apply the $(PIPA)$ algorithm to restore images in Euclidean space $\mathbb{H} = \mathbb{R}^s$, the image restoration algorithm has the following form:

Algorithm 2.2. *Choose the starting point as any x^0, x^1 in $\mathbb{R}^s$.*
Step 1: Calculate

$$w^k = x^k + \alpha_k(x^k - x^{k-1}),$$

with

$$\alpha_k = \begin{cases} \min\left\{\dfrac{\tau_k}{\|x^k - x^{k-1}\|}, \mu_k\right\} & \text{if } \|x^k - x^{k-1}\| \neq 0, \\ \mu_k & \text{otherwise.} \end{cases} \tag{2.7}$$

Step 2: Calculate

$$u_i^k = (1 - \gamma_{k,i})\, w^k + prox_{\epsilon_i f_2} (E - \epsilon_i \nabla f_1) \left(w^k\right).$$

Set $t^k = u_{i_0}^k$, therein $i_0 \in \arg\max \left\{\left\|u_i^k - w^k\right\| : i \in J\right\}$,
Step 3: Calculate

$$x^{k+1} = (1 - \varsigma_k)\, t^k + \varsigma_k \left[t^k - \lambda_k F\left(t^k\right)\right].$$

Set $k = k + 1$ and go back to step 1.

Chapter 3

Relaxed solution mapping method

When studying the mapping of solutions to the variational inequality problem, we develop based on some results on the nonexpansive of Yamada, I., we propose new results on the quasinonexpansive of relaxed mappings of solutions. The mapping of solutions is an extension of the projection on the set C to a projection on a half-space. Then, we propose the relaxed projection algorithm $(RLPA)$ to solve the variational inequality problem, where the constraint domain is the intersection of the fixed point set and the solution set of another variational inequality problem. This result has been published in [CT4] (List of works related to the thesis). Furthermore, we propose the contraction projection algorithm (PCA) to solve the variational inequality problem over the fixed point set of the mapping of solutions $BVI(C, F, G)$ in $\mathbb{R}^n$. These results are published in [CT2]. The application of the $(RLPA)$ and (PCA) algorithms with illustrative calculations and comparisons with other algorithms has been performed on Matlab software.

3.1 Relaxed projection method

3.1.1 Algorithm *(RLPA)*

By combining the conventional projection and the relaxed projection onto the half-space, we construct the algorithm *(RLPA)* to solve the variational inequality problem over fixed point set, which is stated as follows:

Let C be a nonempty, closed, convex subset of a real Hilbert space $\mathbb{H}$ and let the maps $F : \mathbb{H} \to \mathbb{H}$, $G : \mathbb{H} \to \mathbb{H}$ and $\Xi : \mathbb{H} \to \mathbb{H}$.

$$\text{Find } u \in \Omega \text{ such that } \langle G(u), x - u\rangle \geq 0, \quad \forall x \in \Omega, \tag{3.1}$$

where, the set $\Omega = Fix(\Xi) \cap Sol(C, F)$, in which the fixed point set $Fix(\Xi) = \{x \in \mathbb{H} : x = \Xi x\}$, and the set $Sol(C, F)$ is the solution set of problem $VI(C, F)$. The price functions F, G and the sequence of parameters satisfy the following assumptions:

(B_1) The set Ω is non-empty;

(B_2) The price map F is pseudomonotone, L_F-Lipschitz continuous and weakly continuous on C;

(B_3) The price map G is $\beta-$strongly monotone and L_G-Lipschitz continuous;

(B_4) The map Ξ is nonexpansive and satisfies the $I - S$ property demiclosed at 0;

(B_5) For every natural number $k \geq 0$, the positive parameters $\xi_k, \gamma_k, \tau_k, \alpha_k, \tau$ and ν satisfies the following conditions:

$$\begin{cases} \nu \in (0, \min\{1, \frac{1}{L_F}\}), a \in (0, 1 - \nu L_F), \\ b > 0,\ \xi_k \in \left(b, \min\left\{\frac{1}{L_F}, \sqrt{\frac{\nu}{L_F}}\right\}\right), \xi = \lim\limits_{k\to\infty} \xi_k, \\ \gamma_k \in \left(0, \min\left\{\frac{1-\xi_k^2 L_F^2}{2}, \frac{1-\nu L_F - a}{2}\right\}\right), \\ \tau_k \in (c, d) \subset (0, 1),\ \tau \in (0, \frac{2\beta}{L_G^2}), \\ \alpha_k \in (0, 1),\ \lim\limits_{k\to\infty} \alpha_k = 0,\ \sum\limits_{k=0}^{\infty} \alpha_k = \infty. \end{cases}$$

Algorithm 3.1. *(Relaxed projection algorithm (RLPA))*

Choose $x^0 \in \mathbb{H}$, $k = 0$, $\nu > 0$, *choose the positive number sequences* $\{\xi_k\}$, $\{\gamma_k\}$, $\{\tau_k\}$ *and* $\{\alpha_k\}$,

Step 1. Find the projection

$$y^k = \Pi_C[x^k - \xi_k F(x^k)],$$

Step 2. Compute $w^k = x^k - \nu\xi_k F(y^k)$ *and* $t^k = \tau_k z^k + (1 - \tau_k)\Xi z^k$, *where*

$$z^k = \begin{cases} w^k - \frac{d_k}{\|x^k - \xi_k F(x^k) - y^k\|^2}(x^k - \xi_k F(x^k) - y^k) & \text{if } d_k > 0, \\ w^k & \text{otherwise}, \end{cases}$$

with $d_k = \langle x^k - \xi_k F(x^k) - y^k, w^k - y^k\rangle - \gamma_k \|y^k - x^k\|^2$,
Step 3. Compute $x^{k+1} = t^k - \alpha_k \tau G(t^k)$. *Set* $k := k+1$ *and go to Step 1.*

3.1.2 Solution mapping

For each $x \in \mathbb{H}$ and $\xi > 0$, we call $S : \mathbb{H} \to C$ the solution mapping of problem $VI(C,\, F)$, given as follows:

$$Sx = \Pi_C[x - \xi F(x)]. \tag{3.2}$$

We know that $x \in C$ is a solution to problem $VI(C, F)$ if and only if it is a fixed point of the solution mapping S. Given $\gamma > 0$, we call the half-space H_x as follows:

$$H_x = \left\{ w \in \mathbb{H} : \langle x - \xi F(x) - Sx, w - Sx\rangle \le \gamma \|x - Sx\|^2 \right\}. \tag{3.3}$$

Let K be the solution set of problem $VI(C,\, F)$. From the definition of projection Π_C and from (3.2), we have:

$$\langle x - \xi F(x) - Sx, y - Sx\rangle \le 0, \quad \forall y \in C,$$

therefore:

$$\langle x - \xi F(x) - Sx, y - Sx\rangle \le \gamma \|x - Sx\|^2, \quad \forall y \in C.$$

Then, $C \subset H_x$ for all $x \in \mathbb{H}$. On the other hand, for each $z \in \mathbb{H}$, the projection of z on H_x is given in the following explicit form:

$$\Pi_{H_x}(z) = \begin{cases} z - \frac{\langle x - \xi F(x) - Sx, z - Sx\rangle - \gamma \|x - Sx\|^2}{\|x - \xi F(x) - Sx\|^2}(x - \xi F(x) - Sx) & \text{if } z \notin H_x, \\ z \ \text{ otherwise.} \end{cases} \tag{3.4}$$

Next, we define the solution mapping $T : \mathbb{H} \to \mathbb{H}$ of the problem VI(C, F), as follows:

$$Tx = \Pi_{H_x}[x - \nu \xi F(Sx)], \tag{3.5}$$

with parameter $\nu > 0$.

The following lemma shows some important properties of the mappings T and S.

Lemma 3.1. *Suppose the set K is not empty, which is the solution set of problem VI(C, F). We have the following assertions:*

(i) If F is quasi-monotone on K and $L-$Lipschitz-continuous, $\xi \in (0, \frac{1}{L})$, $\gamma \in (0, \frac{1-\xi^2 L^2}{2})$ and $\nu \in \left(\xi^2 L, \min\left\{\frac{1-2\gamma}{L}, 1\right\}\right)$, then T is strongly quasinonexpansive on K. Furthermore, for all $x \in \mathbb{H}$, $x^* \in K$,

$$\begin{aligned}\|Tx - x^*\|^2 \leq & \|x - x^*\|^2 - \nu(1 - \nu L - 2\gamma)\|x - Sx\|^2 \\ & - (\nu - \xi^2 L)\|Tx - Sx\|^2 - (1 - \nu)\|Tx - x\|^2. \quad (3.6)\end{aligned}$$

(ii) If F is $\eta-$strongly inverse quasimonotone on K, then S is strongly nonexpansive quasimonotone on K provided $m \in (\frac{\xi}{2\eta}, 1-2\gamma)$, $\gamma \in (0, \frac{1}{2})$ and $0 < \xi < 2\eta(1 - 2\gamma)$. Furthermore, for all $x \in C$,

$$\begin{aligned}\|Sx - x^*\|^2 \leq & \|x - x^*\|^2 - 2\xi\left(\eta - \frac{\xi}{2m}\right)\|F(x) - F(x^*)\|^2 \\ & - (1 - m - 2\gamma)\|x - Sx\|^2.\end{aligned}$$

(iii) If F is $\zeta-$ strongly quasimonotone on K and $L-$ Lipschitz continuous on $\mathbb{H}$, then S is quasicontractive with constant $\delta := \frac{1}{\sqrt{1+2\xi\zeta-\xi^2 L^2}} \in (0, 1)$, with $\xi \in (0, \frac{2\zeta}{L^2})$.

3.1.3 Convergence theorem

The following lemmas are used to prove the convergence of the iteration sequence in the $(RLPA)$ algorithm.

Lemma 3.2. *Let $\{x^k\}$ and $\{y^k\}$ be two sequences generated by Algorithm 3.1 and let $x^* \in K$. Then, with the assumptions (B_1), (B_2) and (B_4), we have the following assertion:*

$$\begin{aligned}\|z^k - x^*\|^2 \leq & \|x^k - x^*\|^2 - \nu(1 - \nu L_F - 2\gamma_k)\|x^k - y^k\|^2 \\ & - (\nu - \xi_k^2 L_F)\|z^k - y^k\|^2 - (1 - \nu)\|z^k - x^k\|^2. \quad (3.7)\end{aligned}$$

Lemma 3.3. *Suppose the sequence $\{x^k\}$ generated by the algorithm $(RLPA)$ is bounded and $\{x^{k_j}\} \subset \{x^k\}$ is such that $x^{k_j} \rightharpoonup \bar{x}$, satisfies $\lim_{j\to\infty} \|x^{k_j} -$*

$y^{k_j}\| = 0$ *with* $\{y^{k_j}\}$ *being the corresponding subsequence. Then,* $\bar{x} \in Sol(C, F)$.

Theorem 3.1. *Given the cost mapping* F *and* G *satisfying the assumptions* (B_1), $(B_2), (B_3), (B_4)$ *and the parameters satisfying the condition* (B_5), *the two sequences* $\{x^k\}$ *and* $\{y^k\}$ *in the* $(RLPA)$ *Algorithm strongly converge to the unique solution* u *of Problem 3.1.*

3.1.4 Illustrative calculation

3.2 Contraction projection method

3.2.1 Algorithm *(PCA)*

We consider the bilevel variational inequality problem $BVI(C, F, G)$, as follows:

$$\text{Find}\, x^* \in Sol(C, F), \text{ such that } \langle G(x^*), x - x^* \rangle \;\; ge0, \forall x \in Sol(C, F), \tag{3.8}$$

where, the cost mapping $F : C \to \mathbb{R}^n$ is such that $F(x) = Qx + q$, the matrix $Q \in \mathbb{R}^{n \times n}$ is an symmetric matrix, $q \in \mathbb{R}^n$ is a vector, chosen such that F is pseudomonotone, and the constraint domain C is given as follows

$$C = \{x \in \mathbb{R}^n : Ax \geq b\},$$

where, $A \in \mathbb{R}^{m \times n}$ is a matrix and vector $b \in \mathbb{R}^m$. The cost mapping $G : C \to \mathbb{R}^n$ satisfies the condition

(C_1) G is strongly monotone with coefficient $\beta > 0$,

$$\langle G(x) - G(y), x - y \rangle \geq \beta \|x - y\|^2, \quad \forall x, y \in C;$$

(C_2) G is Lipschitz continuous with coefficients $\mathbb{L} > 0$,

$$\|G(x) - G(y)\| \leq \mathbb{L}\|x - y\|, \quad \forall x, y \in C.$$

Algorithm 3.2. *Projection contraction algorithm (PCA)*

Step 1. Given $x^0 \in C, \eta > 0$ *and*

$$\begin{cases} \tau := 1 - \sqrt{1 - \mu(2\beta - \mu\mathbb{L}^2)} \\ \eta > \max\left\{-6\tau_1(Q), \frac{\mathbb{L}\|Q\|(\mathbb{L}+\sqrt{\mathbb{L}^2-\beta^2})}{\beta^2}, \frac{-2\tau_1(Q)(\beta^2+\mathbb{L}^2)}{\beta^2}\right\}, \\ \mu \in \left(0, \frac{2\beta}{\mathbb{L}^2}\right), \alpha_k \in \left(0, \frac{2\mu\beta-2\tau}{\mu^2\mathbb{L}^2-\tau^2}\right), \sum_{k=0}^{\infty} \alpha_k = \infty, \sum_{k=0}^{\infty} \alpha_k^2 < \infty. \end{cases} \tag{3.9}$$

Step 2. $(k = 0, 1, ...)$ *Calculate* y^k

$$\begin{aligned} y^k &= argmin\left\{\frac{1}{2}\langle Qx, x\rangle + \langle q, x\rangle + \frac{\eta}{2}\|x - x^k\|^2 : x \in C\right\}, \\ x^{k+1} &= \Pi_C\left[y^k - \mu\alpha_k G(y^k)\right]. \end{aligned} \tag{3.10}$$

Step 3. Set $k := k + 1$*, go back to Step 2.*

3.2.2 Convergence theorem

The following theorem shows the convergence of the 3.2 Algorithm.

Theorem 3.2. *Suppose the cost functions* F, G *satisfy the assumptions* (C_1), (C_2)*, the sequences* $\{x^k\}$ *and* $\{y^k\}$ *in the Algorithm 3.2 strongly converge to the unique solution* x^* *of the problem* $BVI(C, F, G)$.

3.2.3 Compute error

In this section, we present the computational error of the Algorithm 3.2, applied to solve the problem $BVI(C, F, G)$. At the iteration $k \geq 0$, from the algorithm we calculate the elements y^k and x^{k+1}, assuming

$$\left\|y^k - \operatorname{argmin}\left\{\frac{1}{2}\langle Qx, x\rangle + \langle q, x\rangle + \frac{\eta}{2}\|x - x^k\|^2 : x \in C\right\}\right\| \leq \epsilon, \tag{3.11}$$

$$\left\|x^{k+1} - \Pi_C[y^k - \mu\alpha_k G(y^k)]\right\| \leq \epsilon, \tag{3.12}$$

the ϵ error usually depends on our computer system. When calculating on a computer, the sequences $\{x^k\}$ and $\{y^k\}$ do not necessarily converge to the solution x^* of the problem $BVI(C, F, G)$. For each $\nu > 0$, the element x^k generated by the 3.2 Algorithm is called a $\nu-$solution of the problem $BVI(C, F, G)$ if $\|x^{k+1} - x^k\| \leq \nu$. Therefore, for each number

$\chi > 0$, we define the set Sol_χ as the set of all $\chi-$solutions of the problem $BVI(C, F, G)$,

$$Sol_\chi = \{\bar{x} \in C : \|x^{k+1} - x^k\| \leq \chi\}.$$

Set

$$\begin{cases} \hat{y}^k & = \operatorname{argmin}\left\{\frac{1}{2}\langle Qx, x\rangle + \langle q, x\rangle + \frac{\eta}{2}\|x - x^k\|^2 : x \in C\right\}, \\ \hat{x}^{k+1} & = \Pi_C[\hat{y}^k - \mu\alpha_k G(\hat{y}^k)]. \end{cases}$$

Suppose that there exist $\sigma > 0$ and $\delta > 0$ such that

$$C \subseteq B(x^*, \sigma), \langle G(C), C\rangle := \{\langle G(x), y\rangle : x, y \in C\} \subseteq B(0, \delta).$$

Choose coefficients that satisfy the condition (3.9) and

$$\begin{cases} 0 < \epsilon < \bar{\epsilon}, \bar{\epsilon}^2 > 4\mu\alpha_k\delta + \epsilon^2(1+\tau)^2, \\ \Gamma_\eta := \frac{\eta\bar{\epsilon}^2}{\eta + 2\tau_1(Q)} + \frac{2\tau_1(Q)\sigma^2}{\eta + 2\tau_1(Q)} - 4\mu\alpha_k\delta - \epsilon^2(1+\tau)^2 - 2\epsilon(1+\tau)\sigma. \end{cases} \tag{3.13}$$

Note that $\lim_{\eta\to\infty}\Gamma_\eta = \bar{\epsilon}^2 - 4\mu\alpha_k\delta - \epsilon^2(1+\tau)^2 > 0$, then there exist coefficients satisfying (3.13).

Theorem 3.3. *Suppose that*

(i) $Sol(C, F) \neq \emptyset$;

(ii) $2\mu\alpha_k\delta \leq \epsilon^2$, *conditions* (3.9) *and* (3.13), *and assumptions* $C_1 - C_2$ *are satisfied. Choose a positive number* K *such that*

$$K > \frac{4\sigma^2}{\Gamma_\eta};$$

(iii) The sequences $\{x^k\}$ *and* $\{y^k\}$ *are obtained from the formulas* (3.2)-(3.12).

Then, there exists a natural number $j \in [0, K]$ *such that*

(a) $\|x^j - y^j\| \leq 2\bar{\epsilon}$;

(b) $\|x^k - y^k\| > 2\bar{\epsilon}, \quad \forall k = 0, 1, ..., j-1$;

(c) $x^j \in Sol_{\underline{\epsilon}}$, *here* $\underline{\epsilon} := 2\bar{\epsilon} + 3\epsilon$.

3.2.4 Calculation examples

In this section, we present some computational examples for the (PCA) algorithm, the experiments are programmed on MATLAB R2014a software with PC, Intel(R) Core(TM) i5-7360U CPU @ 2.30GHz 8.00GB Ram. Besides, we compare the efficiency of the (PCA) algorithm with the approximate derivative algorithms $(PVSA)$ of Maingé and the augmented derivative algorithm $(ExtrA)$ of Anh, P.N. et al., in the case where the cost mapping F is pseudomonotone.

Conclusions

The thesis focuses on studying methods for solving variational inequality problems over fixed point sets. New solution algorithms are studied based on iteration methods, projection methods, inertial techniques, principle of auxiliary problem and techniques in Optimization Theory.

1. The main contents of the thesis include

(i) The hybrid inertial contraction algorithm ($HICA$) was developed by us from the iterative technique of Yamada, I. combined with the inertial technique to solve the problem $VIF(\Omega, F)$, where the constraint domain Ω is the fixed point set of the infinite family of demicontractive maps. The advantage of the algorithm is that it only uses the approximate calculation method, does not use projection, contributing to increasing the speed of numerical solution on computers. The results of the algorithm have been published in the work [CT1].

(ii) The inertial parallel approximation algorithm ($PIPA$) is built from inertial techniques and parallel computing, to solve the problem $VIF(\Omega, F)$, with the price map F strongly monotone and Lipschitz continuous, the constraint domain Ω is the fixed point set of m demicontractive maps. We have applied the ($PIPA$) algorithm to the image processing model, to restore images that have been blurred by Gaussian or Motion type. At that time, we have shown the advantages of this algorithm when calculating and comparing with some other image restoration algorithms. The results of this algorithm are published in [CT3].

(iii) The relaxed algorithm ($RLPA$) is built based on the combination of direct projection onto the constraint domain C and relaxed projection onto a half-space. In this algorithm, we have applied finding solutions to the variational inequality problem on the intersection of the fixed point set of a projection and the solution set of another variational inequality. This result is published in [CT4].

(iv) Considering the variational inequality problem on the fixed point set of the solution map $BVI(C, F, G)$, we construct the contraction projection algorithm (PCA) from the direct projection onto the set C and a sub-problem using the DC decomposition technique. The convergence of the iterative sequences to an optimal solution has been shown. The efficiency of the algorithm is calculated through numerical examples and comparison results with other algorithms. The convergence and computational results of the (PCA) algorithm are published in [CT2].

2. Recommendations for further studies

In addition to the results achieved in the thesis, we can research in the following directions:

- Research on algorithms for solving variational inequality problems on fixed point sets, aiming to improve speed and computation time by combining Mann iteration methods, Halpern iteration, and parallel computing techniques. song,...
- Evaluate the error and convergence speed of some algorithms proposed in the thesis, how to choose parameter sets to get better convergence.
- Research new algorithms to relax the conditions placed on price maps, while reducing the projections in each iteration of the algorithm.
- Extend the solution of two-level variational inequality problems to more complex multi-level or constrained domain problems.

The list of works of author related to the Thesis

[CT1] Truong, N.D., Kim, J.K., Anh, T.H.H. (2021), Hybrid inertial contraction projection methods extended to variational inequality problems, *Nonlinear Functional Analysis and Applications* 25(1), pp. 161 - 174. (ISSN:1229 - 1595, Scopus, Q3)

[CT2] Thang, T.V., Anh, P.N., Truong, N.D. (2023), Convergence of the projection and contraction methods for solving bilevel variational inequality problems, *Mathematical Methods in the Applied Sciences* 46(9), 10867 - 10885. (ISSN: 1017 - 1398, SCIE, Q2)

[CT3] Anh, P.N., Gibali, A., Truong, N.D. (2024), Parallel inertial proximal algorithm with applications to image recovery problems, *Journal of Nonlinear and Convex Analysis* 25(11), pp. 2913 - 2931. (ISSN: 1345 - 4773, SCIE, Q2)

[CT4] Anh, P.N., Khanh, P.Q., Truong, N.D. (2024), A relaxed projection method for solving bilevel variational inequality problems, *Optimization*, Doi: 10.1080/02331934.2024.2354456. (ISSN: 0233 - 1934, SCIE, Q2)