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OPTIMALITY CONDITIONS, DUALITY RELATIONS AND DIFFERENTIAL STABILITY IN MULTIOBJECTIVE OPTIMIZATION PROBLEMS

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Introduction

This dissertation presents some new results on optimality conditions, duality relations and the differential stability for some classes of multiobjective optimization problems.

This dissertation consists of four chapters. Chapter 1 presents some basic definitions and facts from Variational Analysis, Convex Analysis and Interval Analysis. Chapter 2 establishes optimality conditions and duality relations for approximate quasi Pareto solutions of nonsmooth semi-infinite interval-valued multiobjective optimization problems. Chapter 3 is devoted to the study of optimality conditions and duality relations for Pareto solutions of fractional interval-valued multiobjective optimization problems with locally Lipschitzian data. Chapter 4 investigates the differential stability of parametric convex multiobjective optimization problems in a finite-dimensional space setting.

The main results of the dissertation include: 1) Establishing necessary and sufficient optimality conditions of Karush–Kuhn–Tucker (KKT) type for approximate quasi Pareto solutions of nonsmooth semi-infinite interval-valued multiobjective optimization problems. 2) Investigating duality relations such as weak, strong and converse-like duality in the sense of Mond–Weir for approximate quasi Pareto solutions of nonsmooth semi-infinite interval-valued multiobjective optimization problems. 3) Providing necessary and sufficient optimality conditions of KKT type for Pareto solutions of fractional interval-valued multiobjective optimization problems with locally Lipschitzian data. 4) Examining duality relations including weak, strong and converse-like duality by way of Mond–Weir for Pareto solutions of fractional interval-valued multiobjective optimization problems. 5) Deriving formulae for computing the subdifferential and the coderivative of the efficient point multifunction of parametric convex multiobjective optimization problems.

Chapter 1

Preliminaries

In this chapter, we provide some basic definitions from Variational Analysis, Interval Analysis and several auxiliary results.

1.1 Basic normal cone

In this section, we recall the definitions of the Fréchet/regular normal cone, the basic / limitting / Mordukhovich normal cone; some calculus rules for the regular normal cone, the basic normal cone to the Cartesian product of two sets, to the intersection of sets; and the presentation of the basic normal cone via the Euclidean projector.

1.2 Subdifferentials

In this section, we recall the definitions of the Fréchet/regular subdifferential, the basic / limitting / Mordukhovich, and the singular subdifferential; some calculus rules for the indicator function, the strictly differentiable function, the subdifferential of a sum/max of finitely many local Lipschitz functions, and the subdifferential of the quotient.

1.3 Coderivatives

In this section, we present the concept of the Fréchet coderivative, the basic / limitting / Mordukhovich coderivative of set-valued mappings and some calculus rules for coderivatives of set-valued mappings.

1.4 Interval and oder relations

In this section, we recall some definitions and properties in Interval Analysis.

Chapter 2

Optimality conditions and duality relations for approximate quasi Pareto solutions in nonsmooth semi-infinite interval-valued multiobjective optimization problems

This chapter deals with approximate solutions of a nonsmooth semi-infinite programming with multiple interval-valued objective functions that has the following form

$$LU\text{-}\mathrm{Min}\, f(x) := (f_1(x), \ldots, f_m(x)) \qquad \text{(SIVP)}$$
$$\text{subject to } \; x \in \mathcal{F} := \{x \in \Omega \, : \, g_t(x) \leq 0, t \in T\},$$

where $f_i \colon \mathbb{R}^n \to \mathcal{K}_c$, $i \in I := \{1, \ldots, m\}$, are interval-valued functions defined by

$$f_i(x) \;=\; [f_i^L(x), f_i^U(x)],$$

in which f_i^L, $f_i^U \colon \mathbb{R}^n \to \mathbb{R}$ are locally Lipschitz functions satisfying $f_i^L(x) \leq f_i^U(x)$ for all $x \in \mathbb{R}^n$ and $i \in I$, $\mathcal{K}_c$ is the class of all closed and bounded intervals in $\mathbb{R}$, i.e.,

$$\mathcal{K}_c = \{[a^L, a^U] \, : \, a^L, a^U \in \mathbb{R}, a^L \leq a^U\},$$

$g_t \colon \mathbb{R}^n \to \mathbb{R}$, $t \in T$, are locally Lipschitz functions, T is an arbitrary set (possibly infinite), and Ω is a nonempty and closed subset of $\mathbb{R}^n$.

The results in this chapter are written based on the paper [CT1].

2.1 Approximate quasi Pareto solutions

Let ϵ_i^L, ϵ_i^U, $i \in I$, be real numbers satisfying $0 \leq \epsilon_i^L \leq \epsilon_i^U$ for all $i \in I$ and put $\mathcal{E} := (\mathcal{E}_1, \ldots, \mathcal{E}_m)$, where $\mathcal{E}_i := [\epsilon_i^L, \epsilon_i^U]$.

Definition 2.1. Let $\bar{x} \in \mathcal{F}$. We say that:

(i) $\bar{x}$ is a *type-1 $\mathcal{E}$-quasi Pareto solution* of (SIVP), denoted by $\bar{x} \in \mathcal{E}\text{-}\mathcal{S}_1^q(\text{SIVP})$, if there is no $x \in \mathcal{F}$ such that

$$\begin{cases} f_i(x) \leq_{LU} f_i(\bar{x}) - \mathcal{E}_i\|x - \bar{x}\|, & \forall i \in I, \\ f_k(x) <_{LU} f_k(\bar{x}) - \mathcal{E}_k\|x - \bar{x}\|, & \text{for at least one } \ k \in I. \end{cases}$$

(ii) $\bar{x}$ is a *type-2 $\mathcal{E}$-quasi Pareto solution* of (SIVP), denoted by $\bar{x} \in \mathcal{E}\text{-}\mathcal{S}_2^q(\text{VP}_p)$, if there is no $x \in \mathcal{F}$ such that

$$\begin{cases} f_i(x) \leq_{LU} f_i(\bar{x}) - \mathcal{E}_i\|x - \bar{x}\|, & \forall i \in I, \\ f_k(x) <_{LU}^s f_k(\bar{x}) - \mathcal{E}_k\|x - \bar{x}\|, & \text{for at least one } \ k \in I. \end{cases}$$

(iii) $\bar{x}$ is a *type-1 $\mathcal{E}$-quasi-weakly Pareto solution* of (SIVP), denoted by $\bar{x} \in \mathcal{E}\text{-}\mathcal{S}_1^{qw}(\text{SIVP})$, if there is no $x \in \mathcal{F}$ such that

$$f_i(x) <_{LU} f_i(\bar{x}) - \mathcal{E}_i\|x - \bar{x}\|, \ \ \forall i \in I.$$

(iv) $\bar{x}$ is a *type-2 $\mathcal{E}$-quasi-weakly Pareto solution* of (SIVP), denoted by $\bar{x} \in \mathcal{E}\text{-}\mathcal{S}_2^{qw}(\text{SIVP})$, if there is no $x \in \mathcal{F}$ such that

$$f_i(x) <_{LU}^s f_i(\bar{x}) - \mathcal{E}_i\|x - \bar{x}\|, \ \ \forall i \in I.$$

It should be note that, if $\mathcal{E} = 0$, i.e., $\epsilon_i^L = \epsilon_i^U = 0$, $i \in I$, then the notion of a type-1 $\mathcal{E}$-quasi Pareto solution (resp., a type-2 $\mathcal{E}$-quasi Pareto solution, a type-1 $\mathcal{E}$-quasi-weakly Pareto solution, a type-2 $\mathcal{E}$-quasi-weakly Pareto solution) defined above coincides with the one of a *type-1 Pareto solution* (resp., a *type-2 Pareto solution*, a *type-1 weakly Pareto solution*, a *type-2 weakly Pareto solution*).

2.2 Optimality conditions

2.2.1 Necessary conditions

Let $\mathbb{R}_+^{|T|}$ denote the set of all functions $\mu\colon T \to \mathbb{R}_+$ taking values $\mu_t := \mu(t) = 0$ for all $t \in T$ except for finitely many points. The active constraint multipliers set at $\bar{x} \in \Omega$ is defined by

$$A(\bar{x}) := \left\{ \mu \in \mathbb{R}_+^{|T|} \ : \ \mu_t g_t(\bar{x}) = 0, \ \ \forall t \in T \right\}.$$

For each $\mu \in A(\bar{x})$, put $T(\mu) := \{t \in T \ : \ \mu_t \neq 0\}$. To obtain the necessary optimality conditions of KKT-type for approximate quasi Pareto solutions of (SIVP), we consider the following constraint qualification condition.

Definition 2.2. Let $\bar{x} \in \mathcal{F}$. We say that $\bar{x}$ satisfies the *limiting constraint qualification* if the following condition holds

$$N(\bar{x}; \mathcal{F}) \subseteq \bigcup_{\mu \in A(\bar{x})} \left[\sum_{t \in T} \mu_t \partial g_t(\bar{x}) \right] + N(\bar{x}; \Omega). \tag{LCQ}$$

It is worth mentioning that thein constraint qualification (LCQ) has been widely used in the literature and it covers almost the existing constraint qualifications of the Mangasarian–Fromovitz and the Farkas–Minkowski types; see e.g.,B.S. Mordukhovich, Variational Analysis and Generalized Differentiation, Vol. 1: Basic Theory, Springer, Berlin, 2006; T.D. Chuong, D.S. Kim, J. Optim. Theory Appl. 160 (2014), 748–762; L.G. Jiao, D.S. Kim, Y. Zhou, Optim. Lett. 15 (2021), 1759–1772.

Theorem 2.1. *Let $\bar{x} \in \mathcal{F}$ and assume that $\bar{x}$ satisfies the* (LCQ). *If $\bar{x} \in \mathcal{E}\text{-}\mathcal{S}_2^{qw}(\text{SIVP})$, then there exist $\lambda^L, \lambda^U \in \mathbb{R}^m_+$ with $\sum_{i \in I}(\lambda_i^L + \lambda_i^U) = 1$, and $\mu \in A(\bar{x})$ such that*

$$0 \in \sum_{i \in I} \left[\lambda_i^L \partial f_i^L(\bar{x}) + \lambda_i^U \partial f_i^U(\bar{x}) \right] + \sum_{t \in T} \mu_t \partial g_t(\bar{x}) + \sum_{i \in I} \left(\lambda_i^L \epsilon_i^U + \lambda_i^U \epsilon_i^L \right) \mathbb{B}_{\mathbb{R}^n} + N(\bar{x}; \Omega). \tag{2.1}$$

2.2.2 Sufficient conditions

Next we present sufficient conditions for approximate quasi Pareto solutions of (SIVP).

In order to obtain these sufficient conditions, we need to introduce concepts of (strictly) generalized convexity at a given point for a family of locally Lipschitz functions. The first definition is inspired from T.D. Chuong, D.S. Kim, J. Optim. Theory Appl. 160 (2014), 748–762.

Definition 2.3. (i) We say that (f, g_T) is *generalized convex on* Ω at $\bar{x} \in \Omega$ if for any $x \in \Omega$, $z_i^{*L} \in \partial f_i^L(\bar{x})$, $z_i^{*U} \in \partial f_i^U(\bar{x})$, $i \in I$, and $x_t^* \in \partial g_t(\bar{x})$, $t \in T$, there exists $\nu \in [N(\bar{x}; \Omega)]^\circ$ satisfying

$$\begin{aligned} & f_i^L(x) - f_i^L(\bar{x}) \geq \langle z_i^{*L}, \nu \rangle, \quad \forall i \in I, \\ & f_i^U(x) - f_i^U(\bar{x}) \geq \langle z_i^{*U}, \nu \rangle, \quad \forall i \in I, \\ & g_t(x) - g_t(\bar{x}) \geq \langle x_t^*, \nu \rangle, \quad \forall t \in T, \\ & \text{and } \ \langle b^*, \nu \rangle \leq \|x - \bar{x}\|, \quad \forall b^* \in \mathbb{B}_{\mathbb{R}^n}. \end{aligned} \tag{2.2}$$

(ii) We say that (f, g_T) is *strictly generalized convex on* Ω at $\bar{x} \in \Omega$ if for any $x \in \Omega \setminus \{\bar{x}\}$, $z_i^{*L} \in \partial f_i^L(\bar{x})$, $z_i^{*U} \in \partial f_i^U(\bar{x})$, $i \in I$, and $x_t^* \in \partial g_t(\bar{x})$, $t \in T$, there exists $\nu \in [N(\bar{x}; \Omega)]^\circ$

satisfying

$$\begin{aligned}
&f_i^L(x) - f_i^L(\bar{x}) > \langle z_i^{*L}, \nu \rangle, \quad \forall i \in I, \\
&f_i^U(x) - f_i^U(\bar{x}) > \langle z_i^{*U}, \nu \rangle, \quad \forall i \in I, \\
&g_t(x) - g_t(\bar{x}) \geq \langle x_t^*, \nu \rangle, \quad \forall t \in T, \\
&\text{and } \ \langle b^*, \nu \rangle \leq \|x - \bar{x}\|, \quad \forall b^* \in \mathbb{B}_{\mathbb{R}^n}.
\end{aligned}$$

Remarrk 2.1. We see that if Ω is convex and f_i^L, f_i^U, $i \in I$, and g_t, $t \in T$, are convex (resp. strictly convex), then (f, g_T) is generalized convex (resp. strictly generalized convex) on Ω at any $\bar{x} \in \Omega$ with $\nu = x - \bar{x}$. Moreover, there exist examples that show the class of generalized convex functions is properly larger than the one of convex functions; see, e.g., T.D. Chuong, D.S. Kim, J. Optim. Theory Appl. 160 (2014), 748–762, Example 3.2 và T.D. Chuong, D.S. Kim, Positivity 20 (2016), 187–207, Example 3.12.

Definition 2.4. (i) We say that (f, g_T) is *$\mathcal{E}$-pseudo-quasi generalized convex on* Ω at $\bar{x} \in \Omega$ if for any $x \in \Omega$, $z_i^{*L} \in \partial f_i^L(\bar{x})$, $z_i^{*U} \in \partial f_i^U(\bar{x})$, $i \in I$, and $x_t^* \in \partial g_t(\bar{x})$, $t \in T$, there exists $\nu \in [N(\bar{x}; \Omega)]^\circ$ satisfying

$$\begin{aligned}
&\langle z_i^{*L}, \nu \rangle + \epsilon_i^U \|x - \bar{x}\| \geq 0 \Rightarrow f_i^L(x) \geq f_i^L(\bar{x}) - \epsilon_i^U \|x - \bar{x}\|, \quad \forall i \in I, \\
&\langle z_i^{*U}, \nu \rangle + \epsilon_i^L \|x - \bar{x}\| \geq 0 \Rightarrow f_i^U(x) \geq f_i^U(\bar{x}) - \epsilon_i^L \|x - \bar{x}\|, \quad \forall i \in I, \\
&g_t(x) \leq g_t(\bar{x}) \Rightarrow \langle x_t^*, \nu \rangle \leq 0, \quad \forall t \in T, \\
&\text{and } \ \langle b^*, \nu \rangle \leq \|x - \bar{x}\|, \quad \forall b^* \in \mathbb{B}_{\mathbb{R}^n}.
\end{aligned} \tag{2.3}$$

(ii) We say that (f, g_T) is *strictly $\mathcal{E}$-pseudo-quasi generalized convex on* Ω at $\bar{x} \in \Omega$ if for any $x \in \Omega \setminus \{\bar{x}\}$, $z_i^{*L} \in \partial f_i^L(\bar{x})$, $z_i^{*U} \in \partial f_i^U(\bar{x})$, $i \in I$, and $x_t^* \in \partial g_t(\bar{x})$, $t \in T$, there exists $\nu \in [N(\bar{x}; \Omega)]^\circ$ satisfying

$$\begin{aligned}
&\langle z_i^{*L}, \nu \rangle + \epsilon_i^U \|x - \bar{x}\| \geq 0 \Rightarrow f_i^L(x) > f_i^L(\bar{x}) - \epsilon_i^U \|x - \bar{x}\|, \quad \forall i \in I, \\
&\langle z_i^{*U}, \nu \rangle + \epsilon_i^L \|x - \bar{x}\| \geq 0 \Rightarrow f_i^U(x) > f_i^U(\bar{x}) - \epsilon_i^L \|x - \bar{x}\|, \quad \forall i \in I, \\
&g_t(x) \leq g_t(\bar{x}) \Rightarrow \langle x_t^*, \nu \rangle \leq 0, \quad \forall t \in T, \\
&\text{and } \ \langle b^*, \nu \rangle \leq \|x - \bar{x}\|, \quad \forall b^* \in \mathbb{B}_{\mathbb{R}^n}.
\end{aligned} \tag{2.4}$$

Remarrk 2.2. By definition, it is easy to see that if (f, g_T) is (strictly) generalized convex on Ω at $\bar{x} \in \Omega$, then for any $\mathcal{E} = (\mathcal{E}_1, \ldots, \mathcal{E}_m)$, where $\mathcal{E}_i = [\epsilon_i^L, \epsilon_i^u]$, $0 \leq \epsilon_i^L \leq \epsilon_i^U$, $i \in I$, (f, g_T) is (strictly) $\mathcal{E}$-pseudo-quasi generalized convex on Ω at $\bar{x} \in \Omega$. Furthermore, the class of (strictly) $\mathcal{E}$-pseudo-quasi generalized convex functions is properly wider than the one of (strictly) generalized convex functions.

Theorem 2.2. *Let $\bar{x} \in \mathcal{F}$ and assume that there exist $\lambda^L, \lambda^U \in \mathbb{R}_+^m$ with $\sum_{i \in I}(\lambda_i^L + \lambda_i^U) = 1$, and $\mu \in A(\bar{x})$ satisfying* (2.1).

(i) *If (f, g_T) is $\mathcal{E}$-pseudo-quasi generalized convex on Ω at $\bar{x}$, then $\bar{x} \in \mathcal{E}$-$\mathcal{S}_2^{qw}$*(SIVP).

(ii) *If (f, g_T) is strictly $\mathcal{E}$-pseudo-quasi generalized convex on Ω at $\bar{x}$, then $\bar{x} \in \mathcal{E}\text{-}\mathcal{S}_1^q(\text{SIVP})$ and we therefore get $\bar{x} \in \mathcal{E}\text{-}\mathcal{S}_2^q(\text{SIVP})$ and $\bar{x} \in \mathcal{E}\text{-}\mathcal{S}_1^{qw}(\text{SIVP})$.*

Remarrk 2.3. (i) By Remark 2.2 and T.Q. Son, N.V. Tuyen, C.-F. Wen, Acta. Math. Vietnam 45 (2020), 435–448, Example 3.2, the condition (2.1) alone is not sufficient to guarantee that $\bar{x}$ is a $\mathcal{E}$-quasi (-weakly) Pareto solution of (SIVP) if the (strict) $\mathcal{E}$-pseudo generalized convexity of (f, g_T) on Ω at $\bar{x}$ is violated.

(ii) If (f, g_T) is $\mathcal{E}$-pseudo-quasi generalized convex on Ω at $\bar{x} \in \mathcal{F}$ and there exist $\lambda^L, \lambda^U \in \mathbb{R}^m_+$ with $\lambda_i^L > 0, \lambda_i^U > 0$, $\forall i \in I$, $\sum_{i\in I}(\lambda_i^L + \lambda_i^U) = 1$, and $\mu \in A(\bar{x})$ satisfying (2.1), then $\bar{x} \in \mathcal{E}\text{-}\mathcal{S}_1^q(\text{SIVP})$.

(iii) Since the class of (strictly) $\mathcal{E}$-pseudo-quasi generalized convex functions is properly wider than the class of (strictly) generalized convex functions, our results in Theorem 2.1 generalize and improve the corresponding results in T.D. Chuong, D.S. Kim, Positivity 20 (2016), 187–207; L.G. Jiao, D.S. Kim, Y. Zhou, Optim. Lett. 15 (2021), 1759–1772; T.Q. Son, N.V. Tuyen, C.-F. Wen, Acta. Math. Vietnam 45 (2020), 435–448; N.V. Tuyen, Investigación Oper. 42 (2021), 223–237.

2.3 Duality Relations

2.3.1 Weak duality

For $y \in \mathbb{R}^n$, $(\lambda^L, \lambda^U) \in \mathbb{R}^m_+ \times \mathbb{R}^m_+ \setminus \{(0,0)\}$, and $\mu \in \mathbb{R}^{|T|}_+$, put

$$\mathcal{L}(y, \lambda^L, \lambda^U, \mu) := f(y) = \left([f_1^L(y), f_1^U(y)], \ldots, [f_m^L(y), f_m^U(y)]\right).$$

In connection with the primal problem (SIVP), we consider the following dual problem in the sense of Mond–Weir (stated in an approximate form):

$$\begin{aligned} &\max_{y\in\Omega} \ \mathcal{L}(y, \lambda^L, \lambda^U, \mu) \\ &\text{s. t. } (y, \lambda^L, \lambda^U, \mu) \in \mathcal{F}_{MW}, \end{aligned} \qquad (\text{SIVD}_{MW})$$

where the feasible set is defined by

$$\mathcal{F}_{MW} := \Big\{(y, \lambda^L, \lambda^U, \mu) \in \Omega \times \mathbb{R}^m_+ \times \mathbb{R}^m_+ \times \mathbb{R}^{|T|}_+ \,:\, 0 \in \sum_{i\in I}[\lambda_i^L \partial f_i^L(y) + \lambda_i^U \partial f_i^U(y)] + \sum_{t\in T} \mu_t \partial g_t(y) + \sum_{i\in I}(\lambda_i^L \epsilon_i^U + \lambda_i^U \epsilon_i^L)\mathbb{B}_{\mathbb{R}^n} + N(y;\Omega), \mu_t g_t(y) \geq 0, t \in T, \sum_{i\in I}(\lambda_i^L + \lambda_i^U) = 1\Big\}.$$

The following theorem describes weak duality relations for approximate quasi Pareto solutions between the primal problem (SIVP) and the dual problem (SIVD_{MW}).

Theorem 2.3 ($\mathcal{E}$-weak duality)**.** *Let $x \in \mathcal{F}$ and $(y, \lambda^L, \lambda^U, \mu) \in \mathcal{F}_{MW}$.*

(i) *If* (f, g_T) *is* $\mathcal{E}$*-pseudo-quasi generalized convex on* Ω *at* y*, then*

$$f(x) \not<^{s}_{LU} \mathcal{L}(y, \lambda^L, \lambda^U, \mu) - \mathcal{E}\|x - y\|.$$

(ii) *If* (f, g_T) *is strictly* $\mathcal{E}$*-pseudo-quasi generalized convex on* Ω *at* y*, then*

$$f(x) \not\leq_{LU} \mathcal{L}(y, \lambda^L, \lambda^U, \mu) - \mathcal{E}\|x - y\|.$$

We can show that the approximate pseudo-quasi generalized convexity of (f, g_T) on Ω used in Theorem 2.3 cannot be omitted.

2.3.2 Strong duality

In this section, we present a theorem that formulates strong duality relations between the primal problem (SIVP) and the dual problem (SIVD$_{MW}$).

Theorem 2.4 ($\mathcal{E}$-strong duality)**.** *Let* $\bar{x}$ *be a type-2* $\mathcal{E}$*-quasi-weakly Pareto solution of* (SIVP) *and assume that the* (LCQ) *holds at this point. Then there exist* $\bar{\lambda}^L, \bar{\lambda}^U \in \mathbb{R}^m_+$*, and* $\bar{\mu} \in A(\bar{x})$ *such that* $(\bar{x}, \bar{\lambda}^L, \bar{\lambda}^U, \bar{\mu}) \in \mathcal{F}_{MW}$*,* $f(\bar{x}) = \mathcal{L}(\bar{x}, \bar{\lambda}^L, \bar{\lambda}^U, \bar{\mu})$*. Furthermore,*

(i) *If* (f, g_T) *is* $\mathcal{E}$*-pseudo-quasi generalized convex on* Ω *at* $\bar{x}$*, then* $(\bar{x}, \bar{\lambda}^L, \bar{\lambda}^U, \bar{\mu})$ *is a type-2* $\mathcal{E}$*-quasi weakly Pareto solution of* (SIVD$_{MW}$)*.*

(ii) *If* (f, g_T) *is strictly* $\mathcal{E}$*-pseudo-quasi generalized convex on* Ω *at* $\bar{x}$*, then* $(\bar{x}, \bar{\lambda}^L, \bar{\lambda}^U, \bar{\mu})$ *is a type-1* $\mathcal{E}$*-quasi Pareto solution of* (SIVD$_{MW}$)*.*

2.3.3 Converse-like duality

We close this section by presenting converse-like duality relations for approximate quasi Pareto solutions between the primal problem (SIVP) and the dual problem (SIVD$_{MW}$).

Theorem 2.5 (Converse-like duality)**.** *Let* $(\bar{x}, \bar{\lambda}^L, \bar{\lambda}^U, \bar{\mu}) \in \mathcal{F}_{MW}$*.*

(i) *If* $\bar{x} \in \mathcal{F}$ *and* (f, g_T) *is* $\mathcal{E}$*-pseudo-quasi generalized convex on* Ω *at* $\bar{x}$*, then* $\bar{x}$ *is a type-2* $\mathcal{E}$*-quasi weakly Pareto solution of* (SIVP)*.*

(ii) *If* $\bar{x} \in \mathcal{F}$ *and* (f, g_T) *is strictly* $\mathcal{E}$*-pseudo-quasi generalized convex on* Ω *at* $\bar{x}$*, then* $\bar{x}$ *is a type-1* $\mathcal{E}$*-quasi Pareto solution of* (SIVP)*.*

Chapter 3

Optimality conditions and duality relations in nonsmooth fractional interval-valued multiobjective optimization

In this chapter, we present results on optimality conditions and duality relations for Pareto solutions of the following fractional multiobjective problem with interval-valued objective functions:

$$LU\text{−Min } F(x) := \left(\frac{f_1(x)}{g_1(x)}, \ldots, \frac{f_m(x)}{g_m(x)}\right) \tag{FIMP}$$
$$\text{s.t. } x \in \Omega := \{x \in S \,:\, h_j(x) \leq 0, j = 1, \ldots, p\},$$

where $f_i, g_i \colon \mathbb{R}^n \to \mathcal{K}_c$, $i \in I := \{1, \ldots, m\}$, are interval-valued functions defined respectively by $f_i(x) = [f_i^L(x), f_i^U(x)]$, $g_i(x) = [g_i^L(x), g_i^U(x)]$, in which f_i^L, f_i^U, g_i^L,$g_i^U : \mathbb{R}^n \to \mathbb{R}$ are locally Lipschitzian functions satisfying $f_i^L(x) \leq f_i^U(x)$ and

$$0 < g_i^L(x) \leq g_i^U(x)$$

for all $x \in S$ and $i \in I$, $\mathcal{K}_c$ is the class of all closed and bounded intervals in $\mathbb{R}$, i.e.,

$$\mathcal{K}_c = \{[a^L, a^U] \,:\, a^L, a^U \in \mathbb{R}, \quad a^L \leq a^U\},$$

$h_j \colon \mathbb{R}^n \to \mathbb{R}$, $j \in J := \{1, \ldots, p\}$, are locally Lipschitzian functions, and S is a nonempty and closed subset of $\mathbb{R}^n$.

The results in this chapter are written based on the paper [CT2].

3.1 Pareto solutions

For the sake of convenience, *we always assume hereafter* that $f_i^L(x) \geq 0$, $\forall x \in S$ and $i \in I$. For each $i \in I$ and $x \in \mathbb{R}^n$, put $F_i(x) := \frac{f_i(x)}{g_i(x)}$. By definition, we have

$$F_i(x) := \frac{f_i(x)}{g_i(x)} = \left[\frac{f_i^L(x)}{g_i^U(x)}, \frac{f_i^U(x)}{g_i^L(x)}\right].$$

Definition 3.1. Let $\bar{x} \in \Omega$. We say that:

(i) $\bar{x}$ is a *type-1 Pareto solution* of (FIMP), denoted by $\bar{x} \in \mathcal{S}_1$(FIMP), if there is no $x \in \Omega$ such that

$$\begin{cases} F_i(x) \leq_{LU} F_i(\bar{x}), & \forall i \in I, \\ F_k(x) <_{LU} F_k(\bar{x}), & \text{for at least one } \ k \in I. \end{cases}$$

(ii) $\bar{x}$ is a *type-2 Pareto solution* of (FIMP), denoted by $\bar{x} \in \mathcal{S}_2$(FIMP), if there is no $x \in \Omega$ such that

$$\begin{cases} F_i(x) \leq_{LU} F_i(\bar{x}), & \forall i \in I, \\ F_k(x) <_{LU}^s F_k(\bar{x}), & \text{for at least one } \ k \in I. \end{cases}$$

(iii) $\bar{x}$ is a *type-1 weakly Pareto solution* of (FIMP), denoted by $\bar{x} \in \mathcal{S}_1^w$(FIMP), if there is no $x \in \Omega$ such that $F_i(x) <_{LU} F_i(\bar{x}), \ \ \forall i \in I$.

(iv) $\bar{x}$ is a *type-2 weakly Pareto solution* of (FIMP), denoted by $\bar{x} \in \mathcal{S}_2^w$(FIMP), if there is no $x \in \Omega$ such that $F_i(x) <_{LU}^s F_i(\bar{x}), \ \ \forall i \in I$.

3.2 Optimality conditions

3.2.1 Necessary conditions

The following result provides a Fritz-John type necessary condition for type-2 weakly Pareto solutions of problem (FIMP).

Theorem 3.1. *If* $\bar{x} \in \mathcal{S}_2^w$(FIMP), *then there exist* $\lambda_i^L \geq 0$, $\lambda_i^U \geq 0$, $i \in I$, *and* $\mu_j \geq 0$, $j \in J$ *with* $\sum_{i\in I}(\lambda_i^L + \lambda_i^U) + \sum_{j\in J}\mu_j = 1$, *such that*

$$0 \in \sum_{i\in I} \frac{\lambda_i^L}{g_i^U(\bar{x})}\left[\partial f_i^L(\bar{x}) - \frac{f_i^L(\bar{x})}{g_i^U(\bar{x})}\partial^+ g_i^U(\bar{x})\right] + \sum_{i\in I} \frac{\lambda_i^U}{g_i^L(\bar{x})}\left[\partial f_i^U(\bar{x}) - \frac{f_i^U(\bar{x})}{g_i^L(\bar{x})}\partial^+ g_i^L(\bar{x})\right]$$
$$+ \sum_{j\in J}\mu_j \partial h_j(\bar{x}) + N(\bar{x}; S), \quad \mu_j h_j(\bar{x}) = 0, \quad j \in J. \tag{3.1}$$

The relation obtained in (3.1) suggests us to define a Karush–Kuhn–Tucker (KKT) type condition when dealing with Pareto solutions of problem (FIMP).

Definition 3.2. Let $\bar{x} \in \Omega$. We say that $\bar{x}$ satisfies the *KKT condition* if (3.1) holds with $\lambda_i^L \geq 0$, $\lambda_i^U \geq 0$, $i \in I$, and $\mu_j \geq 0$, $j \in J$ such that $\sum_{i\in I}(\lambda_i^L + \lambda_i^U) + \sum_{j\in J}\mu_j = 1$ and $(\lambda^L, \lambda^U) \neq (0, 0)$, where $\lambda^L := (\lambda_1^L, \ldots, \lambda_m^L)$ and $\lambda^U := (\lambda_1^U, \ldots, \lambda_m^U)$.

In order to obtain optimality conditions of KKT-type for Pareto solutions of problem (FIMP), we use the following well known constraint qualification.

Definition 3.3. Let $\bar{x} \in \Omega$. We say that the *constraint qualification* (CQ) is satisfied at $\bar{x}$ if there do not exist $\mu_j \geq 0$, $j \in J(\bar{x})$ not all zero, such that

$$0 \in \sum_{j \in J(\bar{x})} \mu_j \partial h_j(\bar{x}) + N(\bar{x}; S), \tag{CQ}$$

where $J(\bar{x}) := \{j \in J \ : \ g_j(\bar{x}) = 0\}$.

It is worth to mentioning here that the above (CQ) reduces to the classical Mangasarian–Fromovitz constraint qualification when the functions $h_1, \ldots, h_p$ are strictly differentiable at such $\bar{x}$ and $S = \mathbb{R}^n$; see e.g., B.S. Mordukhovich, Variational Analysis and Generalized Differentiation, Vol.2: Applications, Springer, Berlin, 2006; T.D. Chuong, N.Q. Huy, J.-C. Yao, SIAM J. Optim. 20 (2009), 1462–1477.

Theorem 3.2. *If $\bar{x} \in \mathcal{S}_2^w(\mathrm{FIMP})$ and the (CQ) holds at $\bar{x}$, then $\bar{x}$ satisfies the KKT condition.*

We can shows that the conclusion of Theorem 3.2 may fail if the (CQ) is not satisfied.

3.2.2 Sufficient conditions

Next we present sufficient conditions for Pareto solutions of (FIMP). In order to obtain these sufficient conditions, we need to introduce concepts of (strictly) generalized convexity at a given point for a family of locally Lipschitzian functions. The following definition is motivated from T.D. Chuong, D.S. Kim, Positivity 20 (2016), 187–207.

Definition 3.4. (i) We say that (F, h) is *generalized convex on* S at $\bar{x} \in S$ if for any $x \in S$, $x_i^{*L} \in \partial f_i^L(\bar{x})$, $x_i^{*U} \in \partial f_i^U(\bar{x})$, $y_i^{*L} \in \partial^+ g_i^L(\bar{x})$, $y_i^{*U} \in \partial^+ g_i^U(\bar{x})$, $i \in I$, and $z_j^* \in \partial h_j(\bar{x})$, $j \in J$, there exists $\nu \in [N(\bar{x}; S)]^\circ$ satisfying

$$\begin{aligned}
&f_i^L(x) - f_i^L(\bar{x}) \geq \langle x_i^{*L}, \nu \rangle, \quad \forall i \in I, \\
&f_i^U(x) - f_i^U(\bar{x}) \geq \langle x_i^{*U}, \nu \rangle, \quad \forall i \in I, \\
&g_i^L(x) - g_i^L(\bar{x}) \leq \langle y_i^{*L}, \nu \rangle, \quad \forall i \in I, \\
&g_i^U(x) - g_i^U(\bar{x}) \leq \langle y_i^{*U}, \nu \rangle, \quad \forall i \in I, \\
&h_j(x) - h_j(\bar{x}) \geq \langle z_j^*, \nu \rangle, \quad \forall j \in J.
\end{aligned}$$

(ii) We say that (F, h) is *strictly generalized convex on* S at $\bar{x} \in S$ if for any $x \in S \setminus \{\bar{x}\}$, $x_i^{*L} \in \partial f_i^L(\bar{x})$, $x_i^{*U} \in \partial f_i^U(\bar{x})$, $y_i^{*L} \in \partial^+ g_i^L(\bar{x})$, $y_i^{*U} \in \partial^+ g_i^U(\bar{x})$, $i \in I$, and $z_j^* \in \partial h_j(\bar{x})$,

$j \in J$, there exists $\nu \in [N(\bar{x}; S)]^{\circ}$ satisfying

$$f_i^L(x) - f_i^L(\bar{x}) > \langle x_i^{*L}, \nu \rangle, \quad \forall i \in I,$$
$$f_i^U(x) - f_i^U(\bar{x}) > \langle x_i^{*U}, \nu \rangle, \quad \forall i \in I,$$
$$g_i^L(x) - g_i^L(\bar{x}) \leq \langle y_i^{*L}, \nu \rangle, \quad \forall i \in I,$$
$$g_i^U(x) - g_i^U(\bar{x}) \leq \langle y_i^{*U}, \nu \rangle, \quad \forall i \in I,$$
$$h_j(x) - h_j(\bar{x}) \geq \langle z_j^{*}, \nu \rangle, \quad \forall j \in J.$$

Remarrk 3.1. We see that if S is convex and f_i^L, f_i^U, $-g_i^L$, $-g_i^U$, $i \in I$, and h_j, $j \in J$, are convex, then (F, h) is generalized convex on S at any $\bar{x} \in S$ with $\nu = x - \bar{x}$. Moreover, the class of generalized convex functions is properly larger than the one of convex functions; see, e.g., T.D. Chuong, D.S. Kim, J. Optim. Theory Appl. 160 (2014), 748–762, Example 3.2 and T.D. Chuong, D.S. Kim, Positivity 20 (2016), 187–207, Example 3.12.

Theorem 3.3. *Let $\bar{x} \in \Omega$ satisfy the KKT condition.*

(i) *If (F, h) is generalized convex on S at $\bar{x}$, then $\bar{x} \in \mathcal{S}_2^w$(FIMP).*

(ii) *If (F, h) is strictly generalized convex on S at $\bar{x}$, then $\bar{x} \in \mathcal{S}_1$(FIMP) and so $\bar{x} \in \mathcal{S}_2$(FIMP) and $\bar{x} \in \mathcal{S}_1^w$(FIMP).*

Remarrk 3.2. The condition (3.1) alone is not sufficient for Pareto solutions of (FIMP) if the (strict) generalized convexity of (F, h) at the point under consideration is violated.

3.3 Duality Relations

For $y \in \mathbb{R}^n$, $(\lambda^L, \lambda^U) \in (\mathbb{R}^m_+ \times \mathbb{R}^m)_+ \setminus \{(0,0)\}$, and $\mu \in \mathbb{R}^p_+$, put

$$\mathcal{L}(y, \lambda^L, \lambda^U, \mu) := F(y) = (F_1(y), \ldots, F_m(y)),$$

where

$$F_i(y) := \frac{f_i(y)}{g_i(y)} = \left[\frac{f_i^L(y)}{g_i^U(y)}, \frac{f_i^U(y)}{g_i^L(y)}\right], \quad i \in I.$$

In connection with the primal problem (FIMP), we consider the following dual problem in the sense of Mond–Weir:

$$LU\!-\max \quad \mathcal{L}(y, \lambda^L, \lambda^U, \mu) \qquad \text{s.t.} \;\; (y, \lambda^L, \lambda^U, \mu) \in \Omega_{MW}, \tag{FIMD$_{MW}$}$$

where the feasible set Ω_{MW} is defined by

$$\Omega_{MW} := \Big\{(y, \lambda^L, \lambda^U, \mu) \in S \times \mathbb{R}^m_+ \times \mathbb{R}^m_+ \times \mathbb{R}^p_+ \;:\; 0 \in \sum_{i \in I} \frac{\lambda_i^L}{g_i^U(y)} \left[\partial f_i^L(y) - \frac{f_i^L(y)}{g_i^U(y)} \partial^+ g_i^U(y)\right]$$
$$+ \sum_{i \in I} \frac{\lambda_i^U}{g_i^L(y)} \left[\partial f_i^U(y) - \frac{f_i^U(y)}{g_i^L(y)} \partial^+ g_i^L(y)\right] + \sum_{j \in J} \mu_j \partial h_j(y) + N(y; S),$$

$$\sum_{j\in J}\mu_j h_j(y) \geq 0, \quad \sum_{i\in I}(\lambda_i^L + \lambda_i^U) + \sum_{j\in J}\mu_j = 1, (\lambda^L, \lambda^U) \neq (0,0)\Big\}.$$

Definition 3.5. Let $(\bar{y}, \bar{\lambda}^L, \bar{\lambda}^U, \bar{\mu}) \in \Omega_{MW}$. We say that

(i) $(\bar{y}, \bar{\lambda}^L, \bar{\lambda}^U, \bar{\mu})$ is a *type-1 Pareto solution* of (FIMD_{MW}), denoted by

$$(\bar{y}, \bar{\lambda}^L, \bar{\lambda}^U, \bar{\mu}) \in \mathcal{S}_1(\text{FIMD}_{MW}),$$

if there is no $(y, \lambda^L, \lambda^U, \mu) \in \Omega_{MW}$ such that $\mathcal{L}(\bar{y}, \bar{\lambda}^L, \bar{\lambda}^U, \bar{\mu}) \preceq_{LU} \mathcal{L}(y, \lambda^L, \lambda^U, \mu)$.

(ii) $(\bar{y}, \bar{\lambda}^L, \bar{\lambda}^U, \bar{\mu})$ is a *type-2 weakly Pareto solution* of (FIMD_{MW}), denoted by

$$(\bar{y}, \bar{\lambda}^L, \bar{\lambda}^U, \bar{\mu}) \in \mathcal{S}_2^w(\text{FIMD}_{MW}),$$

if there is no $(y, \lambda^L, \lambda^U, \mu) \in \Omega_{MW}$ such that $\mathcal{L}(\bar{y}, \bar{\lambda}^L, \bar{\lambda}^U, \bar{\mu}) \preceq_{LU}^s \mathcal{L}(y, \lambda^L, \lambda^U, \mu)$.

3.3.1 Weak duality

The following theorem describes weak duality relations between the primal problem (FIMP) and the dual problem (FIMD_{MW}).

Theorem 3.4 (Weak duality)**.** *Let $x \in \Omega$ and $(y, \lambda^L, \lambda^U, \mu) \in \Omega_{MW}$.*

(i) *If (F, h) is generalized convex on S at y, then*

$$F(x) \not\prec_{LU}^s \mathcal{L}(y, \lambda^L, \lambda^U, \mu).$$

(ii) *If (F, h) is strictly generalized convex on S at y, then*

$$F(x) \not\preceq_{LU} \mathcal{L}(y, \lambda^L, \lambda^U, \mu).$$

Note that the importance of the generalized convexity of (F, h) on S used in Theorem 3.4.This means that the conclusion of Theorem 3.4 may fail if this property has been violated.

3.3.2 Strong duality

Next we present a theorem that formulates strong duality relations between the primal problem (FIMP) and the dual problem (FIMD_{MW}).

Theorem 3.5 (Strong duality)**.** *Suppose that $\bar{x} \in \mathcal{S}_2^w(\text{FIMP})$ and the (CQ) is satisfied at this point. Then there exist $(\bar{\lambda}^L, \bar{\lambda}^U) \in (\mathbb{R}_+^m \times \mathbb{R}_+^m) \setminus \{(0,0)\}$, and $\bar{\mu} \in \mathbb{R}_+^p$ such that $(\bar{x}, \bar{\lambda}^L, \bar{\lambda}^U, \bar{\mu}) \in \Omega_{MW}$ and $F(\bar{x}) = \mathcal{L}(\bar{x}, \bar{\lambda}^L, \bar{\lambda}^U, \bar{\mu})$. Furthermore,*

(i) *If (F, h) is generalized convex on S at $\bar{x}$, then $(\bar{x}, \bar{\lambda}^L, \bar{\lambda}^U, \bar{\mu})$ is a type-2 weakly Pareto solution of (FIMD_{MW}).*

(ii) *If* (F, h) *is strictly generalized convex on* S *at* $\bar{x}$*, then* $(\bar{x}, \bar{\lambda}^L, \bar{\lambda}^U, \bar{\mu})$ *is a type-1 Pareto solution of* (FIMD$_{MW}$)*.*

Remarrk 3.3. The (CQ) condition plays an important role in establishing the strong duality results in Theorem 3.5. This means that if the (CQ) is not satisfied at a type-2 weakly Pareto solution of (FIMP), then strong dual relations in Theorem 3.5 are no longer true at this point.

3.3.3 Converse-like duality

We finish this section by establishing converse-like duality relations for Pareto solutions between the primal problem (FIMP) and the dual one (FIMD$_{MW}$).

Theorem 3.6 (Converse-like duality)**.** *Let* $(\bar{x}, \bar{\lambda}^L, \bar{\lambda}^U, \bar{\mu}) \in \Omega_{MW}$*.*

(i) *If* $\bar{x} \in \Omega$ *and* (F, h) *is generalized convex on* S *at* $\bar{x}$*, then* $\bar{x}$ *is a type-2 weakly Pareto solution of* (FIMP)*.*

(ii) *If* $\bar{x} \in \Omega$ *and* (F, h) *is strictly generalized convex on* S *at* $\bar{x}$*, then* $\bar{x}$ *is a type-1 Pareto solution of* (FIMP)*.*

Chapter 4

Subdifferentials and coderivatives of efficient point multifunctions in parametric convex multiobjective optimization

In this chapter, we aim to study the basic subdifferential as well as the coderivative of the efficient point multifunction for parametric convex multiobjective optimization problems in a finite-dimensional space setting. Namely, by using some advanced tools from convex analysis and variational analysis in finite-dimensional spaces, we obtain, on the one hand, the exact formula for computing the basic subdifferential of the efficient point multifunction of the problem in question which just requires the domination property (Theorem 4.2). This proves that the structures in the finite-dimensional space have their own beauty. Moreover, from this result, we derive a criterion for the Lipschitz-like property with respect to the order cone of the efficient point multifunction (Corollary 4.2). On the other hand, we get formulae for estimating/computing coderivative of the efficient point multifunction as in Theorem 4.3. In the last section, we given some applications to classes of convex multiobjective optimization problems with operator constraints and equilibrium constraints.

The results in this chapter are written based on the paper [CT3].

4.1 Coderivative calculus for convex multifunctions

In the sequel, we will employ the following result on the coderivative of the multifunction H given by $H(x) := H_1(x) \times H_2(x)$, where $H_1 : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$, $H_2 : \mathbb{R}^n \rightrightarrows \mathbb{R}^s$ are multifunctions.

Proposition 4.1. *Let $H_1 : \mathbb{R}^n \rightrightarrows \mathbb{R}^m, H_2 : \mathbb{R}^n \rightrightarrows \mathbb{R}^s$ be two proper convex multifunctions and let $H : \mathbb{R}^n \rightrightarrows \mathbb{R}^m \times \mathbb{R}^s$ be a multifunction defined by $H(x) := H_1(x) \times H_2(x)$. Assume that the* *following qualification condition holds*

$$\operatorname{ri}(\operatorname{dom} H_1) \cap \operatorname{ri}(\operatorname{dom} H_2) \neq \emptyset. \tag{4.1}$$

Then, for any $(\bar{x}, \bar{y}) \in \operatorname{gph} H$ *and* $v \in \mathbb{R}^m \times \mathbb{R}^s$, *we have*

$$D^*H(\bar{x}, \bar{y})(v) = D^*H_1(\bar{x}, \bar{y}_1)(v_1) + D^*H_2(\bar{x}, \bar{y}_2)(v_2), \tag{4.2}$$

where $v = (v_1, v_2)$ *and* $\bar{y} = (\bar{y}_1, \bar{y}_2)$ *with* $\bar{y}_1 \in H_1(\bar{x}), \bar{y}_2 \in H_2(\bar{x})$.

The following result was proved in S. Li, J.-P. Penot, X. Xue, Set Valued Var. Anal. 19 (2011), 505–536, Lemma 49), by passing to the limit of Fréchet coderivatives.

Corollary 4.1. *Suppose that* $H_2 : \mathbb{R}^n \rightrightarrows \mathbb{R}^s$ *is a proper convex multifunction and* $H : \mathbb{R}^n \rightrightarrows \mathbb{R}^n \times \mathbb{R}^s$ *is a multifunction defined by* $H(x) = \{x\} \times H_2(x)$. *Then, for any* $(\bar{x}, (\bar{x}, \bar{y}_2)) \in \operatorname{gph} H$ *and* $(v_1, v_2) \in \mathbb{R}^n \times \mathbb{R}^s$ *we have*

$$D^*H(\bar{x}, (\bar{x}, \bar{y}_2))(v_1, v_2) = v_1 + D^*H_2(\bar{x}, \bar{y}_2)(v_2). \tag{4.3}$$

Definition 4.1. A multifunction $H : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ is called *K-convex* if its epigraph is convex, i.e., for all $x, u \in \mathbb{R}^n$ and $\lambda \in [0, 1]$,

$$\lambda H(x) + (1 - \lambda)H(u) \subset H(\lambda x + (1 - \lambda)u) + K.$$

Clearly, every convex multifunction is also K-convex but not vice versa.

We close this section with a result on the relationship between the basic subdifferential of a K-convex single-valued mapping and the subdifferential (in the sense of convex analysis) of its scalarization.

Proposition 4.2. *If* $h : \mathbb{R}^n \to \mathbb{R}^m$ *is a* K*-convex single-valued mapping and* $\bar{x} \in \mathbb{R}^n$, *then*

$$\partial h(\bar{x}, h(\bar{x}))(v) = \partial \langle v, h \rangle(\bar{x}), \ \forall v \in K^*.$$

In particular, if h *is Fréchet differentiable at* $\bar{x}$, *then*

$$\partial h(\bar{x}, h(\bar{x}))(v) = \nabla h(\bar{x})^*(v), \ \forall v \in K^*,$$

where $\nabla h(\bar{x})^*$ *is the adjoint operator of* $\nabla h(\bar{x})$.

4.2 The basic subdifferential of the efficient point multifunction in convex multiobjective optimization problems

Hereafter, we assume that K is a *proper, pointed, closed and convex cone* in $\mathbb{R}^m$, $f : \mathbb{R}^s \times \mathbb{R}^n \to \mathbb{R}^m$ is a *K-convex single-valued mapping*, and $C : \mathbb{R}^s \rightrightarrows \mathbb{R}^n$ is a *convex multifunction.*

We consider the following *parametric convex multiobjective optimization problem*

$$\text{Min}_K\{f(p,x) \mid x \in C(p)\}, \tag{VP$_p$}$$

where x is a *decision variable*, p is a *perturbation parameter*, f is the *objective mapping*, C is the *constraint mapping*, and the *"minimization"* is understood in conventional terms of multiobjective optimization. More precisely, we say that $\bar{y} \in A$ is an *efficient point* of a given set $A \subset \mathbb{R}^m$ if $A \cap (\bar{y} - K) = \{\bar{y}\}$. The set of all efficient points of A is denoted by $\text{Min}_K A$.

Let $F : \mathbb{R}^s \rightrightarrows \mathbb{R}^m$ be a multifunction defined by

$$F(p) := f(p, C(p)) = \{f(p,x) \mid x \in C(p)\}. \tag{4.4}$$

Then $\bar{y} \in F(p)$ is called an *efficient point of* (VP$_p$) if $\bar{y} \in \text{Min}_K F(p)$. The multifunction $\mathcal{F}$ given by $\mathcal{F}(p) := \text{Min}_K F(p)$ is called the *efficient point multifunction* of (VP$_p$).

The *solution mapping* $\mathcal{S}$ of (VP$_p$) is defined by

$$\mathcal{S}(p) := \{x \in \mathbb{R}^n \mid x \in C(p), f(p,x) \in \mathcal{F}(p)\}.$$

By T. Tanino, SIAM J. Control Optim. 26 (1988), 521–536, Proposition 2.1, F is K-convex multifunction. However, it is worth mentioning here that the efficient point multifunction $\mathcal{F}$ is usually nonconvex.

We now begin with the formulae for computing the subdifferential of the multifunction F given in (4.4).

Theorem 4.1. *Let $(\bar{p}, \bar{x}) \in \text{gph}\,\mathcal{S}$ and $\bar{y} = f(\bar{p}, \bar{x})$. Then, for any $v \in \mathbb{R}^m$, we have*

$$\partial F(\bar{p}, \bar{y})(v) = \bigcup_{(p,u) \in \partial f((\bar{p},\bar{x}),\bar{y})(v)} \Big\{p + D^*C(\bar{p}, \bar{x})(u)\Big\}. \tag{4.5}$$

In particular, if $v \in K^*$, then one has

$$\partial F(\bar{p}, \bar{y})(v) = \bigcup_{(p,u) \in \partial \langle v, f\rangle(\bar{p},\bar{x})} \Big\{p + D^*C(\bar{p}, \bar{x})(u)\Big\}. \tag{4.6}$$

Moreover, if f is Fréchet differentiable at $(\bar{p}, \bar{x})$, then

$$\partial F(\bar{p}, \bar{y})(v) = \nabla_p f(\bar{p}, \bar{x})^*(v) + D^*C(\bar{p}, \bar{x})\big(\nabla_x f(\bar{p}, \bar{x})^*(v)\big), \quad \forall v \in K^*.$$

Remarrk 4.1. In a Banach space setting, the authors in [D.T.V. An, L.T. Tung, (2023). https://doi.org/10.48550/arXiv.2306.06947, Theorem 1] have obtained the formulae for computing the coderivative of $F + K$, i.e., the Fréchet subdifferential of F due to recent results in B.S. Mordukhovich, N.M. Nam, R.B. Rector, T. Tran, Set-Valued Var. Anal. 25 (2017), 731–755. Namely, the authors require two regularity conditions to get (4.5). Here, by the finite-dimensional space setting, we do not need any regularity condition.

Let us establish formulae for computing the subdifferential of $\mathcal{F}$. To do this, we need the domination property for the multifunction F.

Definition 4.2. Let $\bar{p} \in \mathbb{R}^s$. We say that the multifunction F given in (4.4) has the *domination property* around $\bar{p}$ if there exists a neighborhood U of $\bar{p}$ such that

$$F(p) \subseteq \mathcal{F}(p) + K, \quad \forall p \in U.$$

Theorem 4.2. *Let $(\bar{p}, \bar{x}) \in \operatorname{gph} \mathcal{S}$ and $\bar{y} = f(\bar{p}, \bar{x})$. Suppose that F has the domination property around $\bar{p}$. Then, for any $v \in \mathbb{R}^m$ we have*

$$\partial \mathcal{F}(\bar{p}, \bar{y})(v) = \bigcup_{(p,u) \in \partial f((\bar{p},\bar{x}),\bar{y})(v)} \Big\{ p + D^*C(\bar{p}, \bar{x})(u) \Big\}. \tag{4.7}$$

In particular, if $v \in K^*$, then one has

$$\partial \mathcal{F}(\bar{p}, \bar{y})(v) = \bigcup_{(p,u) \in \partial \langle v, f \rangle (\bar{p},\bar{x})} \Big\{ p + D^*C(\bar{p}, \bar{x})(u) \Big\}. \tag{4.8}$$

Moreover, if f is Fréchet differentiable at $(\bar{p}, \bar{x})$, then

$$\partial \mathcal{F}(\bar{p}, \bar{y})(v) = \nabla_p f(\bar{p}, \bar{x})^*(v) + D^*C(\bar{p}, \bar{x})\big(\nabla_x f(\bar{p}, \bar{x})^*(v)\big), \ \forall v \in K^*. \tag{4.9}$$

We end this section with a criterion for the K-Lipschitz-like property of the efficient point multifunction.

Corollary 4.2. *Let $(\bar{p}, \bar{x}) \in \operatorname{gph} \mathcal{S}$ and $\bar{y} = f(\bar{p}, \bar{x})$. Suppose that F has the domination property around $\bar{p}$. Then the efficient point multifunction $\mathcal{F}$ is K-Lipschitz-like around $(\bar{p}, \bar{y})$ if and only if the constraint mapping C is Lipschitz-like around $(\bar{p}, \bar{x})$.*

4.3 Coderivative of F and $\mathcal{F}$ in convex multiobjective optimization problems

In this section, we will present formulae for computing the Fréchet coderivative as well as the basic coderivative of $\mathcal{F}$. It is worth noting here that $\mathcal{F}$ is usually nonconvex. Before presenting the main results of this section, let us review some relevant concepts and results.

Definition 4.3. Let $H \colon \mathbb{R}^s \rightrightarrows \mathbb{R}^m$ be a multifunction and let $(\bar{p}, \bar{y}) \in \operatorname{gph} H$.

(i) The multifunction H is said to be *order semicontinuous at* $(\bar{p}, \bar{y})$ if for any sequence $\{(p_i, y_i)\} \subset \operatorname{epi} H$ converging to $(\bar{p}, \bar{y})$, there exists a sequence $\{(p_i, y_i')\} \subset \operatorname{gph} H$ with $y_i - y_i' \in K$ such that the sequence $\{y_i'\}$ has at least a subsequence converging to $\bar{y}$.

(ii) H is called *order semicontinuous around* $(\bar{p}, \bar{y})$ if there exists a neighborhood U of this point such that H is order semicontinuous at every point $(p, y) \in U \cap \operatorname{gph} H$.

Proposition 4.3. *Let $H : \mathbb{R}^s \rightrightarrows \mathbb{R}^m$ be a multifunction and let $(\bar{p}, \bar{y}) \in \operatorname{gph} H$. Then the following assertions hold:*

(i) *$\widehat{\partial} H(\bar{p}, \bar{y})(v) \subset \widehat{D}^* H(\bar{p}, \bar{y})(v)$ for all $v \in \mathbb{R}^m$. The converse inclusion holds if $v \in K^*_+$ and H is order semicontinuous at $(\bar{p}, \bar{y})$.*

(ii) *if H is order semicontinuous at $(\bar{p}, \bar{y})$, then*

$$\partial H(\bar{p}, \bar{y})(v) \subset D^* H(\bar{p}, \bar{y})(v) \quad \forall v \in \mathbb{R}^m,$$

and the converse inclusion holds if $v \in K_+^*$ and H is order semicontinuous around $(\bar{p}, \bar{y})$.

We are now in a position to establish formulae for computing the Fréchet coderivative and the basic coderivative of $\mathcal{F}$.

Theorem 4.3. *Let $(\bar{p}, \bar{x}) \in \operatorname{gph} \mathcal{S}$ and $\bar{y} = f(\bar{p}, \bar{x})$. Suppose that F has the domination property around $\bar{p}$. Then the following assertions hold:*

(i) *For any $v \in \mathbb{R}^m$, we have*

$$\widehat{D}^* \mathcal{F}(\bar{p}, \bar{y})(v) \supset \bigcup_{(p,u) \in \partial f((\bar{p}, \bar{x}), \bar{y})(v)} \Big\{ p + D^* C(\bar{p}, \bar{x})(u) \Big\}. \tag{4.10}$$

Furthermore, if $v \in K_+^*$ and $\mathcal{F}$ is order semicontinuous at $(\bar{p}, \bar{y})$, then

$$\widehat{D}^* \mathcal{F}(\bar{p}, \bar{y})(v) = \bigcup_{(p,u) \in \partial \langle v, f \rangle (\bar{p}, \bar{x})} \Big\{ p + D^* C(\bar{p}, \bar{x})(u) \Big\}. \tag{4.11}$$

(ii) *If $\mathcal{F}$ is order semicontinuous at $(\bar{p}, \bar{y})$, then for any $v \in \mathbb{R}^m$ we have*

$$D^* \mathcal{F}(\bar{p}, \bar{y})(v) \supset \bigcup_{(p,u) \in \partial f((\bar{p}, \bar{x}), \bar{y})(v)} \Big\{ p + D^* C(\bar{p}, \bar{x})(u) \Big\}. \tag{4.12}$$

Furthermore, if $v \in K_+^*$ and $\mathcal{F}$ is order semicontinuous around $(\bar{p}, \bar{y})$, then one has

$$D^* \mathcal{F}(\bar{p}, \bar{y})(v) = \bigcup_{(p,u) \in \partial \langle v, f \rangle (\bar{p}, \bar{x})} \Big\{ p + D^* C(\bar{p}, \bar{x})(u) \Big\}. \tag{4.13}$$

Remarrk 4.2. In T.D. Chuong, Optim. Lett. 7 (2013), 1087–1117, Proposition 3.5, the authors studied formulae for estimating/computing the Fréchet coderivative and the basic coderivative of $\mathcal{F}$ in the Banach space setting. Besides the domination property of F, the authors employed two additional assumptions related to the mapping S given by

$$S(p, y) = \{x \in \mathbb{R}^n \ : \ x \in C(p), y = f(p, x)\} \tag{4.14}$$

. It is worth emphasizing that we do not need those assumptions anymore in this paper when the spaces in question are finite-dimensional.

4.4 Applications to classes of convex constrained multiobjective optimization problems

4.4.1 Problem with operator constraints

We first consider problem (VP$_p$) where the constraint mapping $C \colon \mathbb{R}^s \rightrightarrows \mathbb{R}^n$ is given in the following form

$$C(p) = \{x \in \mathbb{R}^n \ | \ H(p, x) \cap \Theta \neq \emptyset\}, \tag{4.1}$$

where Θ is a nonempty, closed and convex subset in $\mathbb{R}^l$ and $H\colon \mathbb{R}^s \times \mathbb{R}^n \rightrightarrows \mathbb{R}^l$ is a convex multifunction. The *inverse image* of Θ under the mapping H is defined by

$$H^{-1}(\Theta) = \{(p,x) \in \mathbb{R}^s \times \mathbb{R}^n \ \mid \ H(p,x) \cap \Theta \neq \emptyset\}.$$

Then, it is easy to see that $\operatorname{gph} C = H^{-1}(\Theta)$. We now claim that C given by (4.1) is a convex multifunction.

Proposition 4.4. *The constraint mapping C given in* (4.1) *is convex.*

The following result gives formulae for computing the subdifferential of $\mathcal{F}$.

Theorem 4.4. *Let $(\bar p, \bar x) \in \operatorname{gph} \mathcal{S}$ and $\bar y = f(\bar p, \bar x)$. Suppose that F has the domination property around $\bar p$ and the following qualification condition holds*

$$\operatorname{ri}(\operatorname{rge} H) \cap \operatorname{ri}(\Theta) \neq \emptyset. \tag{4.2}$$

Then, for any $v \in \mathbb{R}^m$ we have

$$\partial\mathcal{F}(\bar p, \bar y)(v) = \bigcup_{(p,u)\in\partial f((\bar p,\bar x),\bar y)(v)} \Big\{ p+w \mid (w,-u) \in D^*H((\bar p,\bar x,\bar w))(N(\bar w;\Theta)) \Big\}, \tag{4.3}$$

for any $\bar w \in H(\bar p, \bar x) \cap \Theta$. Furthermore, if $v \in K^*$, then one has

$$\partial\mathcal{F}(\bar p, \bar y)(v) = \bigcup_{(p,u)\in\partial\langle v,f\rangle(\bar p,\bar x)} \Big\{ p+w \mid (w,-u) \in D^*H((\bar p,\bar x,\bar w))(N(\bar w;\Theta)) \Big\}.$$

The following result is a direct consequence of Theorem 4.4.

Corollary 4.3. *Let $(\bar p, \bar x) \in \operatorname{gph} \mathcal{S}$ and $\bar y = f(\bar p, \bar x)$. Suppose that F has the domination property around $\bar p$ and that h is concave with respect to Θ^∞ and strictly differentiable at $(\bar p, \bar x)$. If the following qualification condition holds*

$$\operatorname{ri}(\operatorname{rge} h) \cap \operatorname{ri}(\Theta) \neq \emptyset, \tag{4.4}$$

then, for any $v \in \mathbb{R}^m$ we have

$$\partial\mathcal{F}(\bar p, \bar y)(v) = \bigcup_{(p,u)\in\partial f((\bar p,\bar x),\bar y)(v)} \Big\{ p+w \mid (w,-u) \in \nabla h(\bar p,\bar x)^* N(\bar w;\Theta) \Big\},$$

where $\bar w := h(\bar p, \bar x)$. Furthermore, if $v \in K^*$, then one has

$$\partial\mathcal{F}(\bar p, \bar y)(v) = \bigcup_{(p,u)\in\partial\langle v,f\rangle(\bar p,\bar x)} \Big\{ p+w \mid (w,-u) \in \nabla h(\bar p,\bar x)^* N(\bar w;\Theta) \Big\}.$$

4.4.2 Problem with equilibrium constraints

In the last subsection, we consider problem (VP_p) that involves *equilibrium constraints* of the type

$$C(p) = \{x \in \mathbb{R}^n \ \mid \ 0 \in g(p,x) + Q(p,x)\}, \tag{4.5}$$

where $g\colon \mathbb{R}^s \times \mathbb{R}^n \to \mathbb{R}^l$ is a single-valued mapping and $Q\colon \mathbb{R}^s \times \mathbb{R}^n \rightrightarrows \mathbb{R}^l$ is a multifunction. Systems of the form (4.5) are widely recognized for their effectiveness in describing sets of optimal solutions to parameter-dependent variational and related problems.

Problem (VP_p) with the constraint given in (4.5) are usually called multiobjective optimization problems with equilibrium constraints, see, e.g., B.S. Mordukhovich, Variational Analysis and Applications, Springer, Switzerland, 2018. In our setting, we assume that g and Q *are convex*. Then, by putting

$$h(p,x) := (p,x,-g(p,x)) \quad \text{and} \quad \Theta := \operatorname{gph} Q,$$

we obtain

$$\operatorname{rge} h = \operatorname{gph}(-g) \quad \text{and} \quad \operatorname{gph} C = \{(p,x) \in \mathbb{R}^s \times \mathbb{R}^n \mid h(p,x) \in \Theta\} = h^{-1}(\Theta).$$

The constraint mapping C is convex due to the convexity of g and Q.

The final result of this subsection presents a formula for computing the subdifferential of the efficient point multifunction of (VP_p) with the constraint in the form of (4.5).

Theorem 4.5. *Let $(\bar{p},\bar{x}) \in \operatorname{gph} \mathcal{S}$ and $\bar{y} = f(\bar{p},\bar{x})$. Suppose that F has the domination property around $\bar{p}$. If the following qualification condition holds*

$$\operatorname{ri}(\operatorname{gph}(-g)) \cap \operatorname{ri}(\operatorname{gph} Q) \neq \emptyset, \tag{4.6}$$

then, for any $v \in \mathbb{R}^m$ we have

$$\begin{aligned}\partial \mathcal{F}(\bar{p},\bar{y})(v) = \bigcup_{(p,u)\in\partial f((\bar{p},\bar{x}),\bar{y})(v)} \Big\{ & p + w \mid \exists z \in \mathbb{R}^l \ \ \text{with} \\ & (w,-u) \in D^*Q(\bar{p},\bar{x},-g(\bar{p},\bar{x}))(z) + D^*g(\bar{p},\bar{x})(z) \Big\}.\end{aligned} \tag{4.7}$$

General Conclusions

The main results of the dissertation include:

1) Establishing necessary and sufficient optimality conditions of Karush-Kuhn-Tucker (KKT) type for approximate quasi Pareto solutions of nonsmooth semi-infinite interval-valued multiobjective optimization problems.

2) Investigating duality relations such as weak, strong and converse-like duality in the sense of Mond–Weir for approximate quasi Pareto solutions of nonsmooth semi-infinite interval-valued multiobjective optimization problems.

3) Providing necessary and sufficient optimality conditions of KKT type for Pareto solutions of fractional interval-valued multiobjective optimization problems with locally Lipschitzian data.

4) Examining duality relations including weak, strong and converse-like duality by way of Mond–Weir for Pareto solutions of fractional interval-valued multiobjective optimization problems.

5) Deriving formulae for computing the subdifferential and the coderivative of the efficient point multifunction of parametric convex multiobjective optimization problems.

Some further research directiosn are as follow:

- Optimality conditions (first-order and higher-order) and duality relations for multiobjective optimal control problems with interval data;

- Optimality conditions (first-order and higher-order) and duality relations for robust optimization problems;

- The stability/directional differential stability for parametric multiobjective optimization problems;

- The existence of solutions for optimization problems with uncertain data.

List of Author's Related Papers

[CT1] N.H. Hung, H.N. Tuan, N.V. Tuyen, On approximate quasi Pareto solutions in nonsmooth semi-infinite interval-valued vector optimization problems, Appl. Anal. 102 (2023), 2432–2448. (SCIE - Q2)

[CT2] N.H. Hung, N.V. Tuyen, Optimality conditions and duality relations in nonsmooth fractional interval-valued multiobjective optimization, Appl. Set-Valued Anal. Optim. 5 (2023), No. 1, 31-47, 31-47. (SCOPUS - Q3)

[CT3] D.T.V. An, N.H. Hung, N.V. Tuyen, Subdifferentials and coderivatives of efficient point multifunctions in parametric convex vector optimization, J. Optim. Theory Appl. 202 (2024), 745-770. (SCI - Q1)

The main results of this dissertation have been presented at

- The weekly seminar of Department of Analysis, Faculty of Mathematics, Hanoi Pedagogical University 2;
- The weekly seminar of Variational Analysis Group, Faculty of Mathematics, Hanoi Pedagogical University 2;
- 2022 International Online Workshop on Optimization Theory and Applications, January 19–21, 2022, KimLab for Applied Mathematics, Busan, Korea;
- Miniworkshop on Optimization and Control Theory, December 23–24, 2022, Optimization and Control Department, Institute of Mathematics (VAST).