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LONG TIME BEHAVIOR AND CONTROL PROBLEM FOR SOME CLASSES OF STRONGLY DEGENERATE PARABOLIC EQUATIONS

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INTRODUCTION

1. Motivation and history of the problem

Many processes in nature, science, technology and engineering lead to the study of classes of parabolic equations, such as heat transfer processes, diffusion processes, models in population ecology, etc. Therefore, the study of this equations class has important meaning in science and technology. That is why it has attracted widespread attention. One of those approaches is to study the long-time behavior of the solutions when time is infinity, as it allows us to understand and predict the future dynamics, since we can make the appropriate adjustments to achieve the desired results. Besides, the study of controllability of classes of parabolic equations is also very important because under the influence of control function classes, we can control the systems to desired states.

In recent years, the existence and qualitative properties of solutions, in particular, long time behavior and controllability have been studied for many classes of parabolic equations. For example, the class of semilinear parabolic equations in the non-degenerate or weakly degenerate case is studied by many authors in both bounded and unbounded domains (see P. Cannarsa, P. Martinez and J. Vancostenoble (2008); V. V. Chepyzhov, M. I. Vishik (2002);E. Fernández-Cara and S. Guerrero (2006); M. X. Thao (2016); N. T. C. Thuy and N. M. Tri (2002); C. K. Zhong, M. H. Yang and C.Y. Sun (2006),...). Up to now, the results on global attractor theory, controllability theory for the class of non-degenerate parabolic equations are plentiful and quite complete. However, the corresponding results in the case of strongly degenerate parabolic equations are not many, because the strong degeneracy of the system has caused great difficulties in terms of mathematics. For example, we lack of necessary embedding theorems, necessary results for regularity of solution, the results on the extreme principle, necessary Carleman-type estimates.

The problem of controllability for parabolic equations has attracted the interest of several authors in the past two decades. This theory is now well understood for uniformly parabolic equations in both bounded and unbounded domains. In the last decade, the controllability of degenerate parabolic equations has been studied widely by many authors. However, most of existing results related to degenerate parabolic equations in dimension one (see P. Cannarsa, G. Fragnelli and J. Vancostenoble (2006); P. Cannarsa, P. Martinez and J. Vancostenoble (2005); L. Miller (2005); C. Wang and R. Du (2013) and references therein). To the best of our knowledge, there is only a few results for degenerate parabolic equations in multi-dimensional domains involving Grushin operator (see C.T. Anh and V.M. Toi (2013); C.T. Anh and V.M. Toi (2016); K. Beauchard, P. Cannarsa and R. Guglielmi (2014); A. Koenig (2017); J. P. Raymond and H. Zidani (1999); E. Zuazua (1997)), Kolmogorov-type equations (see K. Beauchard (2013, 2014); J. Le Rousseau and I. Moyano (2016)), and boundary-degenerate parabolic equations (see P. Cannarsa, P. Martinez and J. Vancostenoble (2009); C. Wang and R. Du (2013)).

Strong degenerate parabolic equations occur naturally in physics, chemistry, biology, etc. Currently, the study of long-time behavior and controllability of strongly degenerate parabolic equations that are an open, meaningful and interesting problem, which has attracted attention of many mathematicians around the world.

In conclusion, although there have been some recent results on the actraction theory and controllability for the class of strongly degenerate parabolic equations, however, the results obtained are still few and there are many open issues. Therefore, we choose these issues to study in the thesis entitled "**Long time behavior and control problem for some classes of strongly degenerate parabolic equations**"

2. Purpose of the thesis

To study the existence, long-time behavior of solutions and control problem for some classes of strongly degenerate parabolic equations using the methods of Functional Analysis.

3. Object and scope of the thesis

- *Research object*: The existence, long-time behavior of solutions and control problem for some classes of strongly degenerate parabolic equations.
- *Research scope:*

  Content 1: To study the existence and long-time behavior of semilinear parabolic equation involving the strongly degenerate operator on bounded domain.

  Content 2: To study the existence and long-time behavior of semilinear parabolic equation involving the strongly degenerate operator $P_{s,\gamma}$ on $\mathbb{R}^N$.

  Content 3: To sutdy controllability problem for parabolic equation involving the strongly degenerate operator in some multidimensional domains.

4. Research methods

- *To study the existence of solutions*: Galerkin approximation, the compactness and energy methods.
- *To study the existence attractors:* Infinite dimensional dissipative dynamical systems theory methods.
- *In order to study the controllability of the linear problem:* The Hilbert Uniqueness Method (HUM).

5. Results of thesis

The thesis obtains the following main results:

- Proving the existence and uniqueness of weak solution, the existence of the global attractors for the class of semilinear parabolic equation involving the strong degenerate operator Δ_λ on the bounded domain.
- Proving the existence and uniqueness of weak solution, the existence of the global attractors for the class of semilinear parabolic equation involving the strong degenerate operator $P_{s,\gamma}$ on $\mathbb{R}^N$.
- For the parabolic equation involving the strong degenerate operator $P_{s,\gamma}$ in multi–dimensional case: We proved that the null controllability in any time $T > 0$ holds when $s+\gamma \in (0, 1/2)$ (weak degeneracy). When $s = \gamma = 1/2$ (strong degeneracy), we proved that the null controllability holds in large time. We have proved the null controllability in any time $T > 0$ when $s + \gamma > 1$ (too strong degeneracy).

6. Structure of the thesis

Beside Introduction, Conclusion, Author's works related to the thesis and References, the thesis includes 4 chapters:

• Chapter 1. Preliminaries.

• Chapter 2. Global attractor for the class of semilinear strong degenerate parabolic equation on the bounded domain.

• Chapter 3. Global attractor for the class of semilinear strong degenerate parabolic equation on whole space.

• Chapter 4. Controllability of the class of strong degenerate parabolic equation.

Chapter 1

PRELIMINARIES

In this chapter, we present some preliminaries including: Operators; Some functional spaces; controllable theory of linear parabolic equation; some additional results (the usual inequalities, the compactness methods, the weak form of the bounded convergent theorem) that is used to prove the main results of the thesis in the following chapters.

1.1. Operators

In this section, we introduce the operator classes studied in the problems of the thesis: The Δ_λ-Laplace operator was introduced by Franchi and Lanconelli in 1982 and more recently 2013 was developed by A. E. Kogoj and E. Lancenolli; The strongly degenerate operator $P_{s,\gamma}$ is an extension of the Grushin operator. This operator is degenerate on two intersecting surfaces $x = 0$ and $y = 0$, and considered by Thuy and Tri (2012).

1.2. The function spaces

In this section, we list some function spaces that will be used in the thesis: Lebesgue spaces $L^p(\Omega), 1 \leq p < +\infty$, $L^\infty(\Omega)$,$L^2(\Omega)$; time dependent function spaces $C([0,T];X)$, $L^p(0,T;X), 1 \leq p \leq +\infty$; weighted Sobolev spaces $\overset{\circ}{W}{}^{1,2}_\lambda(\Omega)$, $D(\Delta_\lambda)$, $S^1(\mathbb{R}^N)$ and $S^2(\mathbb{R}^N)$.

1.3. Global attractor

In this section, we recall some of the concepts of dynamics, global attractors, and theorems about the existence of global attractors to be used in the thesis. The content of this section is based on monographs by J. C. Robinson (2001) and R. Temam (1997).

1.4. Controllable theory of linear parabolic system

1.4.1. Some definitions

In this subsection, we present some usual definitions involving the controllability problem in infinite dimensional including: exact controllability, exact controllability to the trajectory, null controllability and approximate controllability.

1.4.2. The Hilbert Uniqueness Method (HUM)

In this subsection, we present the Hilbert Uniqueness Method (HUM for sort) which was first introduced by J.-L. Lions (1988) to study the controllability of linear system in infinite dimensional space.

1.5. Some usual results

In this section, we recall some of the primary but important inequalities as well as some important propositions and theorems that are frequently used in the thesis: Aubin–Lions–Simon compact lemma by F. Boyer and P. Fabrie (2013); Lemma 6.1 on the strong convergence of nonlinear functions by P. G. Geredeli (2015); Bessel–Parseval equality; spherical coordinates formula in $\mathbb{R}^N, N \geq 3$.

Chapter 2

GLOBAL ATTRACTOR FOR THE CLASS OF SEMILINEAR STRONG DEGENRATE PARABOLIC EQUATION ON THE BOUNDED DOMAIN

In this chapter, we study a class of semilinear strongly degenerate parabolic equation involving the strong degeneracy operator Δ_λ on the bounded domain $\Omega \subset \mathbb{R}^N, N \geq 2$, with a new class of nonlinearities that are unbounded for growth conditions.

The contents of this chapter is written based on the paper [CT1] in the section of author's works related to the thesis that has been published.

2.1. Problem setting

In this section, we consider the following semilinear strongly degenerate parabolic equation

$$\begin{cases} u_t - \Delta_\lambda u + f(u) = g(x), & x \in \Omega, t > 0, \\ u(x,t) = 0, & x \in \partial\Omega, t > 0, \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases} \tag{2.1}$$

where Ω is a bounded domain in $\mathbb{R}^N$ $(N \geq 2)$ with smooth boundary $\partial\Omega$. To study problem (2.1), we assume that the initial datum $u_0 \in L^2(\Omega)$ is given, the nonlinearity f and the external force g satisfy the following conditions:

(F) $f : \mathbb{R} \to \mathbb{R}$ is a continuously differentiable function satisfying

$$f'(u) \geq -\ell, \tag{2.2}$$

$$f(u)u \geq -\mu u^2 - C_1, \tag{2.3}$$

where C_1, ℓ are two positive constants, $0 < \mu < \gamma_1$ with $\gamma_1 > 0$ is the first eigenvalue of the operator $-\Delta_\lambda$ in Ω with the homogeneous Dirichlet boundary condition;

(G) $g \in L^2(\Omega)$.

2.2. Existence and Uniqueness of Weak Solutions

Definition 2.1. A function u is called a weak solution of problem (2.1) on $(0,T)$ if $u \in C([0,T];L^2(\Omega)) \cap L^2(0,T;\overset{\circ}{W}{}^{1,2}_\lambda(\Omega))$, $f(u) \in L^1(Q_T), u(0) = u_0$, $\frac{du}{dt} \in L^2(0,T;(\overset{\circ}{W}{}^{1,2}_\lambda(\Omega))^*) + L^1(Q_T)$ and

$$\frac{du}{dt} - \Delta_\lambda u + f(u) = g \text{ in } L^2(0,T;(\overset{\circ}{W}{}^{1,2}_\lambda(\Omega))^*) + L^1(Q_T)$$

or equivalently,

$$\left\langle \frac{du}{dt} - \Delta_\lambda u + f(u), w \right\rangle = \langle g, w \rangle$$

for all test functions $w \in W := \overset{\circ}{W}{}^{1,2}_\lambda(\Omega) \cap L^\infty(\Omega)$ and for a.e. $t \in (0,T)$.

The result of the existence and uniqueness of weak solution of problem (2.1) are presented in the following theorem.

Theorem 2.1. *Assume **(F)-(G)** hold. Then for any $u_0 \in L^2(\Omega)$ and $T > 0$ given, problem* (2.1) *has a unique weak solution u on the interval $(0,T)$. Moreover, the mapping $u_0 \mapsto u(t)$ is continuous on $L^2(\Omega)$, that is, the solutions depend continuously on the initial data.*

2.3. Existence of a global attractor

By Theorem 2.1, we can define a continuous (nonlinear) semigroup $S(t) : L^2(\Omega) \to L^2(\Omega)$ associated to problem (2.1) as follows

$$S(t)u_0 := u(t),$$

where $u(\cdot)$ is the unique weak solution of (2.1) with the initial datum $u_0 \in L^2(\Omega)$.

2.3.1. Existence of bounded absorbing sets

Lemma 2.1. *Let hypotheses **(F)-(G)** hold. Then the semigroup $\{S(t)\}_{t\geq 0}$ has a bounded absorbing set in $L^2(\Omega)$.*

Lemma 2.2. *The semigroup $\{S(t)\}_{t\geq 0}$ has a bounded absorbing set in $\overset{\circ}{W}{}^{1,2}_{\lambda}(\Omega)$.*

2.3.2. Asymptotic compactness of $\{S(t)\}_{t\geq 0}$

Lemma 2.3. *The semigroup $\{S(t)\}_{t\geq 0}$ has a bounded absorbing set in $D(\Delta_\lambda)$.*

Lemma 2.4. *The embedding $D(\Delta_\lambda) \hookrightarrow \overset{\circ}{W}{}^{1,2}_{\lambda}(\Omega)$ is compact.*

Using Lemma 2.2, Lemma 2.3 and Lemma 2.4, we get the main result of this section.

Theorem 2.2. *Suppose* **(F)–(G)** *hold. Then the semigroup $S(t)$ generated by problem* (2.1) *has a compact global attractor $\mathscr{A}$ in the space $\overset{\circ}{W}{}^{1,2}_{\lambda}(\Omega)$.*

Comments on the chapter. To end this chapter, now we give a a number of comment on the results of the problem.

- The Δ_λ-Laplace operator contains many degenerate elliptic operators such as the Grushin type operator and the strongly degenerate operator of the form $P_{s,\gamma}$. The operator Δ_λ has strong degeneracy, so when studying the class of parabolic equation involving this operator, we have to build weighted Sobolev spaces $\overset{\circ}{W}{}^{1,2}_{\lambda}(\Omega)$, $D(\Delta_\lambda)$ and prove that embedding $D(\Delta_\lambda) \hookrightarrow \overset{\circ}{W}{}^{1,2}_{\lambda}(\Omega)$ is compact.

- The main difference compared with the previous works is that the nonlinear term $f(u)$ only belongs to $L^1(Q_T)$ due to no restriction imposed on its upper growth. This introduces some essential difficulties when establishing *a priori* estimates and passing to the limit for the nonlinear term.

Conclusion of Chapter 2.

In this chapter, we have presented the results on the existence, uniqueness and continuous dependence of the solution on the initial data, the existence of the global attractor along with its smoothness. To prove the existence and uniqueness of solutions, we used Galerkin approximation and compact methods. To prove the smoothness of the global attractor, we use the asymptotic *a priori* estimate method.

Chapter 3

GLOBAL ATTRACTOR FOR A CLASS OF SEMILINEAR STRONGLY DEGENERATE PARABOLIC EQUATIONS ON WHOLE SPACE

In this chapter, we study a class of semilinear parabolic equations involving the strongly degenerate operator $P_{s,\gamma}$ on $\mathbb{R}^N, N \geq 2$.

The contents of this chapter is written based on the paper [CT2] in the section of author's works related to the thesis that has been published.

3.1. Problem setting

In this chapter, we consider the following semilinear strongly degenerate parabolic equation

$$\begin{cases} \dfrac{\partial u}{\partial t} - P_{s,\gamma}u + f(X,u) + \lambda u = g(X), & X \in \mathbb{R}^N, t > 0, \\ u(X,0) = u_0(X), & X \in \mathbb{R}^N, \end{cases} \tag{3.1}$$

where $\lambda > 0, X = (x, y, z), \mathbb{R}^N = \mathbb{R}^{N_1} \times \mathbb{R}^{N_2} \times \mathbb{R}^{N_3} (N_1, N_2, N_3 \geq 1)$, and $P_{s,\gamma}$ being the strongly degenerate operator which introduced in Chapter 1.

To study problem (3.1), we assume that the initial datum $u_0 \in L^2(\mathbb{R}^N)$ is given, the nonlinearity f and the external force g satisfy the following conditions:

(F) $f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is a continuously differentiable function satisfying

$$f'_u(X,u) \geq -\ell, \tag{3.2}$$

$$f(X,u)u \geq -\mu u^2 - C_1(X), \tag{3.3}$$

where $\ell > 0$, $0 < \mu < \lambda$, $C_1(\cdot) \in L^1(\mathbb{R}^N) \cap L^2(\mathbb{R}^N)$ is nonnegative function;

(G) $g \in L^2(\mathbb{R}^N)$.

3.2. Existence and uniqueness of weak solutions

Definition 3.1. A function u is called a weak solution of problem (3.1) on the interval $(0, T)$ if $u \in C([0, T]; L^2(\mathbb{R}^N)) \cap L^2(0, T; S^1(\mathbb{R}^N)), u(0) = u_0$, and

$$\langle u_t, w\rangle - \langle P_{\alpha,\beta} u, w\rangle + \langle f(X, u), w\rangle + \lambda\langle u, w\rangle = \langle g, w\rangle,$$

for all test functions $w \in S^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ and for a.e. $t \in (0, T)$.

The result of the existence and uniqueness of weak solution of problem (3.1) are presented in the following theorem.

Theorem 3.1. *Assume* **(F)-(G)** *hold. Then for any $u_0 \in L^2(\mathbb{R}^N)$ and $T > 0$ given, problem* (3.1) *has a unique weak solution u on the interval* $(0, T)$*. Moreover, the mapping $u_0 \mapsto u(t)$ is continuous on $L^2(\mathbb{R}^N)$.*

3.3. Existence of global attractor

By Theorem 3.1, we can define a continuous semigroup $S(t) : L^2(\mathbb{R}^N) \to L^2(\mathbb{R}^N)$ associated to problem (3.1) as follows

$$S(t)u_0 := u(t),$$

where $u(\cdot)$ is the unique global weak solution of (3.1) with the initial datum u_0. We will prove that the semigroup $S(t)$ has a global attractor $\mathscr{A}$ in the spaces $L^2(\mathbb{R}^N)$ and $S^1(\mathbb{R}^N)$.

3.3.1. Existence of bounded absorbing sets

Lemma 3.1. *The semigroup $\{S(t)\}_{t\geq 0}$ has a bounded absorbing set in $L^2(\mathbb{R}^N)$.*

Lemma 3.2. *The semigroup $\{S(t)\}_{t\geq 0}$ has a bounded absorbing set in $S^1(\mathbb{R}^N)$.*

Lemma 3.3. *Suppose* $(\mathbf{F})-(\mathbf{G})$ *hold. Then for every bounded subset B in $L^2(\mathbb{R}^N)$, there exists a constant $T = T(B) > 0$ such that*

$$\|u_t(s)\|^2_{L^2(\mathbb{R}^N)} \leq \rho_3 \text{ for all } u_0 \in B, \text{ and } s \geq T,$$

where $u_t(s) = \dfrac{d}{dt}(S(t)u_0)|_{t=s}$ *and* ρ_3 *is a positive constant independent of* B.

We now show the existence of a bounded absorbing set in $S^2(\mathbb{R}^N)$.

Lemma 3.4. *The semigroup* $\{S(t)\}_{t\geq 0}$ *has a bounded absorbing set in* $S^2(\mathbb{R}^N)$*, i.e., there exists a constant* $\rho_4 > 0$ *such that for any bounded subset* $B \subset L^2(\mathbb{R}^N)$*, there is a* $T_B > 0$ *such that*

$$\|P_{\alpha,\beta}u(t)\|^2_{L^2(\mathbb{R}^N)} + \|u(t)\|^2_{L^2(\mathbb{R}^N)} \leq \rho_4, \text{ for any } t \geq T_B, u_0 \in B.$$

3.3.2. Existence of a global attractor in $L^2(\mathbb{R}^N)$

Lemma 3.5. *Suppose* **(F) - (G)** *hold. Then for any* $\epsilon > 0$ *and any bounded subset* $B \subset L^2(\mathbb{R}^N)$*, there exist* $T = T(\epsilon, B) > 0$ *and* $K = K(\epsilon, B) > 0$ *such that for all* $t \geq T$ *and* $R \geq K$,

$$\int_{\mathbb{R}^N \setminus B_R^*} |u(X,t)|^2 dX \leq \epsilon.$$

Now, we show the asymptotic compactness of $S(t)$ in $L^2(\mathbb{R}^N)$.

Lemma 3.6. *Suppose that* **(F) - (G)** *hold. Then* $S(t)$ *is asymptotically compact in* $L^2(\mathbb{R}^N)$*, that is, for any bounded sequence* $\{x_n\} \subset L^2(\mathbb{R}^N)$ *and any sequence* $t_n \geq 0, t_n \to \infty$, $\{S(t_n)x_n\}$ *has a convergent subsequence with respect to the topology of* $L^2(\mathbb{R}^N)$.

We are now ready to prove the existence of a global attractor in $L^2(\mathbb{R}^N)$.

Theorem 3.2. *Suppose that* **(F) - (G)** *hold. Then the semigroup* $S(t)$ *generated by problem* (3.1) *has a global attractor* $\mathscr{A}_{L^2}$ *in* $L^2(\mathbb{R}^N)$.

3.3.3. Existence of a global attractor in $S^1(\mathbb{R}^N)$

Lemma 3.7. *Suppose* **(F) - (G)** *hold. Then for any* $\epsilon > 0$ *and any bounded subset* $B \subset L^2(\mathbb{R}^N)$*, there exist* $T = T(\epsilon, B) > 0$ *and* $K = K(\epsilon, B) > 0$ *such that for all* $t \geq T$ *and* $R \geq K$,

$$\int_{\mathbb{R}^N \setminus B_R^*} |\nabla_{\alpha,\beta} u|^2 dX \leq \epsilon.$$

Now, we show the asymptotic compactness of $S(t)$ in $S^1(\mathbb{R}^N)$.

Lemma 3.8. *Suppose that* **(F) - (G)** *hold. Then $S(t)$ is asymptotically compact in $S^1(\mathbb{R}^N)$, that is, for any bounded sequence $\{x_n\} \subset S^1(\mathbb{R}^N)$ and any sequence $t_n \geq 0, t_n \to \infty$, $\{S(t_n)x_n\}$ has a convergent subsequence with respect to the topology of $S^1(\mathbb{R}^N)$.*

We are now ready to prove the existence of a global attractor for $S(t)$ in $S^1(\mathbb{R}^N)$.

Theorem 3.3. *Suppose that* **(F) - (G)** *hold. Then the semigroup $S(t)$ generated by problem* (3.1) *has a global attractor $\mathscr{A}_{S^1}$ in $S^1(\mathbb{R}^N)$.*

Comments on the chapter. To end this chapter, now we give some comments on the mail results of Chapter 3.

- The nonlinear class $f(X,u)$: we can remove the assumption on the upper bound of the growth rate, which is widely used in many papers (see C.T. Anh and T. D. Ke (2009); C.T. Anh and L. T. Tuyet (2013); A. E. Kogoj and S. Sonner (2013, 2014); D. Li and C. Sun (2016); M. X. Thao (2016); P. T. Thuy and N. M. Tri (2013) with bounded domain and C.T. Anh (2014); C.T. Anh and L. T. Tuyet (2013) with unbounded domain). In particular, the above class of nonlinearities is very large, it contains both the Sobolev type growth condition and the polynomial type growth condition, and even exponential growth such as $f(X,u) = e^u$.

- The results in this chapter are still valid when $\mathbb{R}^N$ is replaced by an arbitrary unbounded domain Ω (then we need to add a homogeneous Dirichlet boundary condition on $\partial\Omega$). The difference of this chapter from Chapter 2 is that the embeddings are not compact (so the semigroup $S(t)$ generated by the problem is not compact) and this causes great difficulties when studying the existence of experience and the existence of a global attractor. To overcome these difficulties, we used Aubin–Lions–Simon Compact Lemma to prove the existence of a solution, and combined the tail estimates method introduced by B. Wang in 1999 and the asymptotic *a priori* estimate method to prove the asymptotic compactness of the semigroup $S(t)$ in $L^2(\mathbb{R}^N)$.

Conclusion of Chapter 3

In this chapter, we proved the existence of a global attractors in the spaces $L^2(\mathbb{R}^N)$ and $S^1(\mathbb{R}^N)$ for the semigroup generated by the problem (3.1). First, using the Galerkin method, we prove the existence of a global weak solution and then construct a semigroup generated by the problem (3.1). Next, in order to overcome the difficulties caused by the non-compact of the embedding, we combined the tail estimates method introduced by B. Wang in 1999 and the asymptotic *a priori* estimate method.

Chapter 4

CONTROLLABILITY OF A CLASS OF STRONGLY DEGENERATE PARABOLIC EQUATION

In this chapter, we study the null controllability of parabolic equation involving the strongly degenerate operator $P_{s,\gamma}$ in some muti-dimensional cases. At the first of chapter, we introduce problem and state the main result. And then, we prove some auxiliary results: well-posedness of the problem, Fourier decomposition, dissipation speed, and especially a new Carleman inequality. Next, using the HUM method, Fourier decomposition, estimates of dissipation speed and new Carleman inequality which just established, the proof of the controllability is reduced to the uniform observability with respect to Fourier frequencies of the adjoint system after Fourier decomposition.

The contents of this chapter is written based on the paper [CT3] in the section of author's works related to the thesis that has been published.

4.1. Problem setting and statement of the main result

In this chapter, we are interested in the null controllability of the following strongly degenerate parabolic equation

$$\begin{cases} u_t - \Delta_x u - \Delta_y u - |x|^{2s}|y|^{2\gamma}\Delta_z u = v(x,y,z,t)1_\omega, & (x,y,z,t) \in \Omega \times (0,T), \\ u = 0, & (x,y,z,t) \in \partial\Omega \times (0,T), \\ u(x,y,z,0) = u_0(x,y,z), & (x,y,z) \in \Omega, \end{cases} \tag{4.1}$$

where $\Omega := \Omega_{12} \times \Omega_3$, Ω_{12} is a smooth bounded open subset of $\mathbb{R}^{N_1+N_2}$ such that $(0_{\mathbb{R}^{N_1}}, 0_{\mathbb{R}^{N_2}}) \in \Omega_{12}$, Ω_3 is a smooth bounded open subset of $\mathbb{R}^{N_3}$; $(x,y,z) = (x_1,\ldots,x_{N_1},y_1,\ldots,y_{N_2},z_1,\ldots,z_{N_3}) \in \mathbb{R}^{N_1+N_2} \times \mathbb{R}^{N_3}$; $\omega \subset \Omega$ and $s,\gamma \geq 0, s+\gamma > 0$; 1_ω denotes the characteristic function of an open non-empty subset ω of Ω.

We say that problem (4.1) is null controllable (in time T) if for every $u_0 \in L^2(\Omega)$ given, there exists $v \in L^2(\omega \times (0,T))$ such that (4.1) has solution $u(x,y,z,t)$ satisfying $u(\cdot,\cdot,\cdot,T) = 0$.

The aim of this chapter is to prove the following result.

Theorem 4.1. *Let $\omega = \omega_{12} \times \Omega_3$, where ω_{12} be an empty open subset of Ω_{12}.*

1. *If $s+\gamma \in (0,1/2)$, then system (4.1) is null controllable in any time $T > 0$.*
2. *If $s = \gamma = 1/2$, then there exists $T^* > 0$ such that system (4.1) is null controllable in time $T \geq T^*$.*
3. *If $s+\gamma > 1$, then system (4.1) is not null controllable.*

4.2. Some auxiliary results

4.2.1. Well-posedness of problem

Using the Galerkin approximate method, we get the following well–posedness.

Theorem 4.2. *For any $u_0 \in L^2(\Omega)$ and $v \in L^2(0,T;L^2(\omega))$ given, problem (4.1) has a unique weak solution satisfying*

$$u \in C([0,T];L^2(\Omega)) \cap L^2(0,T;S_0^1(\Omega)).$$

Moreover,

$$\|u(t)\|^2_{L^2(\Omega)} + \|u\|^2_{L^2(0,T;S_0^1(\Omega))} \leq C\left(\|u_0\|^2_{L^2(\Omega)} + \|v\|^2_{L^2(0,T;L^2(\omega))}\right),$$

where C is a positive constant independing on u_0 and v.

4.2.2. Fourier decomposition and dissipation speed

Let $(\chi_n)_{n\in\mathbb{N}^*}$ be the non-decreasing sequence of eigenvalues of the operator $-\Delta_z$ in $H^2(\Omega_3) \cap H_0^1(\Omega_3)$ and the associated eigenvalue vectors $(\varphi_n(y))_{n\in\mathbb{N}^*}$, that is,

$$\begin{cases} -\Delta_z \varphi_n(z) = \chi_n \varphi_n(z), & z \in \Omega_3, \\ \varphi_n(z) = 0, & z \in \partial\Omega_3. \end{cases}$$

For any weak solution $u(x,y,z,t)$ of (4.1) and any control $v(x,y,z,t)$, we set

$$u_n(x,y,t) = \int_{\Omega_3} u(x,y,z,t)\varphi_n(z)dz; \quad v_n(x,y,t) = \int_{\Omega_3} v(x,y,z,t)\varphi_n(z)dz. \quad (4.2)$$

then by substituting (4.2) into (4.1), we obtain the following

Proposition 4.1. *Let $u_0 \in L^2(\Omega)$ be given and let u be the corresponding unique weak solution of (4.1). Then, for every $n \in \mathbb{N}^*$, the function $u_n(x,y,t)$ is the unique weak solution of the problem*

$$\begin{cases} \dfrac{\partial u_n}{\partial t} - \Delta_x u_n - \Delta_y u_n + \chi_n |x|^{2s}|y|^{2\gamma} u_n = v_n 1_{\omega_{12}}(x,y) & \textit{trong } \Omega_{12} \times (0,T), \\ u_n = 0 & \textit{trÃłn } \partial\Omega_{12} \times (0,T), \\ u_n(x,y,0) = u_{0,n}(x,y) & \textit{trong } \Omega_{12}, \end{cases} \quad (4.3)$$

where $u_{0,n}(x,y) = \int_{\Omega_3} u_0(x,y,z)\varphi_n(z)dz$.

We know that the smallest eigenvalue of $-\Delta\varphi(x,y) + \chi_n|x|^{2s}|y|^{2\gamma}\varphi(x,y)$ in $H^2(\Omega_{12}) \cap H_0^1(\Omega_{12})$ is given by

$$\lambda_{n,s,\gamma} := \min_{\substack{\varphi \in H_0^1(\Omega_{12}) \\ \varphi \neq 0}} \left\{ \frac{\int_{\Omega_{12}} \left(|\nabla\varphi(x,y)|^2 + \chi_n |x|^{2s}|y|^{2\gamma}|\varphi|^2 \right) dxdy}{\int_{\Omega_{12}} |\varphi|^2 dxdy} \right\}.$$

Then, the asymptotic (when $|n| \to +\infty$) of $\lambda_{n,s,\gamma}$ is given in the following proposition.

Proposition 4.2. *For every $s,\gamma \geq 0, s+\gamma > 0$, there exist $c_* = c_*(s,\gamma) > 0$ and $c^* = c^*(s,\gamma) > 0$ such that*

$$c_* \chi_n^{\frac{1}{1+s+\gamma}} \leq \lambda_{n,s,\gamma} \leq c^* \chi_n^{\frac{1}{1+s+\gamma}} \quad \forall n \in \mathbb{N}^*. \quad (4.4)$$

4.2.3. Carleman inequality

Theorem 4.3. *(Carleman inequality). There exist positive constants $\mathcal{K}_1 = \mathcal{K}_1(\beta)$ and $\mathcal{K}_2 = \mathcal{K}_2(\beta)$ such that any $w \in C^0([0,T]; L^2(\Omega_{12})) \cap L^2(0,T; H_0^1(\Omega_{12}))$ satisfies*

$$\mathcal{K}_2 \iint\limits_{\Omega_{12}\times(0,T)} e^{-2M\sigma} \left(\frac{M}{t(T-t)} |\nabla w|^2 + \frac{M^3}{(t(T-t))^3} |w|^2 \right) dx dy dt$$
$$\leq \iint\limits_{\omega_{12}\times(0,T)} e^{-2M\sigma} \frac{M^3}{(t(T-t))^3} |w|^2 dx dy dt + \iint\limits_{Q} e^{-2M\sigma} |G_{n,s,\gamma} w|^2 dx dy dt, \quad (4.5)$$

where

$$M = M(\chi_n, T, \beta) := \begin{cases} \mathcal{K}_1 \max\left\{T + T^2; \chi_n^{2/3} T^2\right\} & \text{if } 0 < s+\gamma < 1/2, \\ \mathcal{K}_1 \max\left\{T + T^2; \chi_n^{1/2} T^2\right\} & \text{if } s = \gamma = 1/2. \end{cases}$$

4.3. Proof of main result

4.3.1. Strategy for the proof of Theorem 4.1

By the HUM method, the null controllability of problem (4.1) is equivalent to an observability inequality for the adjoint problem

$$\begin{cases} w_t + \Delta_x w + \Delta_y w + |x|^{2s}|y|^{2\gamma} \Delta_z w = 0, & (x,y,z,t) \in \Omega \times (0,T), \\ w = 0, & (x,y,t) \in \partial\Omega \times (0,T), \\ w(x,y,z,T) = w_T(x,y,z), & (x,y,z) \in \Omega. \end{cases} \quad (4.6)$$

Definition 4.1. System (4.6) is observable in ω in times T in times $C > 0$, such that, for every $w_T \in L^2(\Omega)$, the solution w of (4.6) satisfies

$$\|w(x,y,z,0)\|^2_{L^2(\Omega)} \leq C \int_0^T \int_\omega |w(x,y,z,t)|^2 dx dy dt.$$

Let w be the solution of (4.6). We can develop $w = w(x,y,z,t)$ in Fourier series in z:

$$w_n(x,y,t) = \int_{\Omega_3} w(x,y,z,t)\varphi_n(z)dz; \quad w_{T,n}(x,y) = \int_{\Omega_3} w_T(x,y,z)\varphi_n(z)dz.$$

Then $w_n(x,y,t)$ is the solution of the following problem (adjoint problem of (4.3)).

$$\begin{cases} \partial_t w_n + \Delta_x w_n + \Delta_y w_n - \chi_n |x|^{2s}|y|^{2\gamma} w_n = 0, & (x,y,t) \in \Omega_{12} \times (0,T), \\ w_n = 0, & (x,y,t) \in \partial\Omega_{12} \times (0,T), \\ w_n(x,y,T) = w_{T,n}(x,y), & (x,y) \in \Omega_{12}, \end{cases} \tag{4.7}$$

We note that, for almost everywhere $t \in (0,T)$, and for any open set ω_{12} of Ω_{12}, we have

$$\int_{\omega_{12}\times\Omega_3} |w(x,y,z,t)|^2 dxdydz = \sum_n \int_{\omega_{12}} |w_n(x,y,t)|^2 dxdy.$$

This is the so-called Bessel-Parseval equality. Thus, to prove the observability of system (4.6), it is sufficient to study the observability of system (4.7) uniformly with respect to $n \in \mathbb{N}^*$.

Definition 4.2. (Uniform observability). Let ω_{12} be an open subset of Ω_{12}. System (4.7) is observable in ω_{12} uniformly with respect to $n \in \mathbb{N}^*$ if there exists $C > 0$ such that, for every $n \in \mathbb{N}^*, w_{T,n} \in L^2(\Omega_{12})$, the solution of (4.7) satisfies

$$\int_{\Omega_{12}} |w_n(x,y,0)|^2 dxdy \leq C \int_0^T \int_{\omega_{12}} |w_n(x,y,t)|^2 dxdydt.$$

4.3.2. Observability inequality

The proof of the following theorem relies on the Carleman estimate established in Theorem 4.3.

Theorem 4.4. *Let ω_{12} be a nonempty open subset of Ω_{12}.*

- *If $s + \gamma \in (0, 1/2)$ then system (4.7) is observable in ω_{12} uniformly with respect to $n \in \mathbb{N}^*$ for any $T > 0$.*
- *If $s = \gamma = 1/2$ then there exists a time $T^* > 0$ such that system (4.7) is observable in ω_{12} uniformly with respect to $n \in \mathbb{N}^*$ for $T \geq T^*$.*

4.3.3. End of the proof of Theorem 4.1

We now prove the following negative result and this will complete the proof of Theorem 4.1.

Theorem 4.5. *Let Ω_{12} be the form $(-1,1)^{N_1+N_2}$. If $s+\gamma > 1$, and $\omega_{12} = (a,b)^{N_1+N_2}$, with*

$$\left[\frac{\min\{(N_1-1)^{\frac{s+1}{2}} N_2^{\frac{\gamma+1}{2}}, (N_2-1)^{\frac{\gamma+1}{2}} N_1^{\frac{s+1}{2}}\}}{N_1^{\frac{s+1}{2}} N_2^{\frac{\gamma+1}{2}}}\right]^{\frac{1}{2+s+\gamma}} < a < b \leq 1,$$

then system (4.7) *is not observable in ω_{12} uniformly with respect to $n \in \mathbb{N}^*$.*

The proof relies on the choice of particular test functions, that falsify uniform observability.

Comments on the chapter. In this chapter, we have used the proof scheme as in C.T. Anh and V.M. Toi (2013); K. Beauchard, P. Cannarsa and R. Guglielmi (2014) and especially in K. Beauchard, P. Cannarsa and M. Yamamoto (2014) for the Grushin operator. However, the strong degeneracy of operator $P_{s,\gamma}$ leads to the presence of term $\chi_n|x|^{2s}|y|^{2\gamma}w$ in the expression of operator $P_{n,s,\gamma}$, which causes some difficulties in establishing the Carleman inequality. To overcome this difficulty, besides choosing an appropriate weight function σ and the constant λ, we exploit several techniques used in the proof of Lemma 5.2 in E. Fernández–Cara (1997) and Proposition 2.5 in C.T. Anh and V.M. Toi (2013). The obtained results are an extension of the existing results (see C.T. Anh and V.M. Toi (2013); K. Beauchard, P. Cannarsa and R. Guglielmi (2014); K. Beauchard, P. Cannarsa and M. Yamamoto (2014)).

Conclusion of Chapter 4

In this chapter, we study the null controllability of parabolic equation involving the strongly degenerate operator $P_{s,\gamma}$ in $\Omega \subset \mathbb{R}^N$. The main results are:

- When $s+\gamma \in (0, 1/2)$, we proved that the null controllability in any time $T > 0$ holds.
- When $s = \gamma = 1/2$, we proved that the null controllability holds in large time.
- When $s+\gamma > 1$, we proved that the null controllability fails in any time $T > 0$.

CONCLUSION AND RECOMMENDATION

1. Results of the thesis

In this thesis, we study the existence of solutions, long-time behavior and null controllability for some class of strongly degenerate parabolic equations. The results of the thesis include:

- Proving the existence and uniqueness of the weak solution, the existence of a global attractor for a class of semilinear parabolic equation involving the strongly degenerate operator Δ_λ on the bounded domain.
- Proving the existence and uniqueness of the weak solution, the existence of the global attractors for a class of semilinear parabolic equation involving the strongly degenerate operator $P_{s,\gamma}$ on $\mathbb{R}^N$.
- For the parabolic equation involving the strongly degenerate operator $P_{s,\gamma}$ in multi–dimensional case: We proved that the null controllability in any time $T > 0$ holds when $s + \gamma \in (0, 1/2)$ (weak degeneracy). When $s = \gamma = 1/2$ (strong degeneracy), we proved that the null controllability holds in large time. We have proved the null controllability in any time $T > 0$ when $s+\gamma > 1$ (too strong degeneracy).

2. Recommendation

Beside the results obtained in this thesis, some open questions need to study as follows:

- Study the properties of the global attractor obtained in Chapters 2 and 3, such as studying the smoothness of the attractor, evaluating the number of fractal dimensions, the continuous dependence on the parameter,...

- Study the existence and uniqueness of the weak solution, the existence of a global attractor for a class of semilinear parabolic equation involving the strongly degenerate operator Δ_λ on $\mathbb{R}^N$.

- Study the null controllability of problem (4.1) in the remaining cases of s and γ: Is it null controllable when $s+\gamma \in (1/2;1)$? Is it null controllable at large enough time, not null controllable at small time, when $s+\gamma=1, s,\gamma \neq 1/2$?

AUTHOR'S WORKS RELATED TO THE THESIS THAT HAVE BEEN PUBLISHED

[CT1]. D.T. Quyet, L.T. Thuy and N.X. Tu (2017), Semilinear strongly degenerate parabolic equations with a new class of nonlinearities, *Vietnam J. Math.* 45 (3), 507–517.

[CT2]. N.X. Tu (2021), Global attractor for a semilinear strongly degenerate parabolic equation with exponential nonlinearity in unbounded domains, *Commun. Korean Math. Soc.,* accepted.

[CT3]. C.T. Anh and N.X. Tu, Null controllability of a strongly degenerate parabolic equation, *submitted.*

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- Seminar of Department of Analysis, Faculty of Mathematics, Hanoi National University of Eduacation;
- Seminar of the Department of Mathematics, Faculty of Natural Sciences, Hung Vuong University.