MINISTRY OF EDUCATION AND TRAINING
HANOI PEDAGOGYCAL UNIVERSITY NO.2

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BEHAVIOR OF SOLUTIONS FOR SOME CLASSES OF NONLOCAL EVOLUTION EQUATIONS

SUMMARY OF DOCTORAL THESIS IN MATHEMATICS

Major: Mathematical Analysis
Code: 9 46 01 02

HA NOI, 2021

This dissertation has been written at Hanoi Pedagogical University No.2

Supervisor:

Referee 1: ....
....

Referee 2: ....
....

Referee 3: ....
....

The thesis shall be defended at the University level Thesis Assessment Council at Hanoi Pedagogical University No.2 on ....

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INTRODUCTION

1. Motivation and history of the problem

In the theory of differential equation, the terminology "nonlocal differential equation" is employed to indicate differential equations in which the derivative of state functions is defined not only at point-wise but determined by an integral formulation (it is also said to be "derivative with memory"). A typical class of nonlocal differential equations given below is utilized to describe anomalous diffusion phenomena

$$\partial_t\big(k * [u - u(0)]\big) = \Delta u, \tag{1}$$

where $u = u(t, x)$ is state function, the kernel $k \in L^1_{loc}(\mathbb{R}^+)$, '*' denotes the Laplace convolution, i.e., $(k * v)(t) = \int_0^t k(t-s)v(s)ds$ and Δ is the Laplace operator. This class of differential equations have been studied by many mathematicians. Readers can see some typical results on the anomalous diffusion equations in the works given by Ke (2020), Kemppainen (2016), Pozo (2019), Vergara (2015). In particular, when

$$k(t) = g_{1-\alpha}(t) = \frac{t^{-\alpha}}{\Gamma(1-\alpha)}, t > 0, \alpha \in (0, 1), \tag{2}$$

equation (1) is the subdiffusion equation, which has been the subject of study by many mathematicians over the past two decades. Equation (1) with the kernel k given by (2) is in the form of fractional differential equations (FrDEs). It is obviously that the fractional derivative equations are excellent models for nonlocal differential equations, they have been attracted many attentions of researcher in recent years.

FrDE is the research direction of fractional analysis proposed in 1695 by Leibniz and Euler then developed by many mathematicians such as Laplace, Fourier, Liouville, Riemann, Laurant, Hardy, and Riesz,..., see Kilbas (2006), Miller (1993), Sabatier (2007) and Podlubny (1999). In the past few decades, many applications of fractional analysis in general and FrDE in particular have been found in the science and technology disciplines, such as problems related to electrochemical chemistry, rheology, porous materials, elastic materials, fractal materials,... Details of some problems described by FrDE can be found in monographs (see Hilfer (2000), Sabatier (2007), Shantanu. (2011) and Podlubny (1999)). FrDE's growing range of applications has prompted a lot of qualitative research in recent years.

One of the most important questions in the study of integro-differential equations is the behavior of solutions. In the scope of this thesis, behavior of solutions of NDE includes questions about short-time behavior of solutions, the stability and the existence of decay solutions.

In the past two decades, the stability question for FrDE in finite and infinite dimensional space has received much attention by many domestic and foreign mathematicians. Concerning finite dimensional FrDEs, the problem of studying the stability has achieved many basic and systematic results. The Lyapunov function method to study stability for FrDE was proposed by

Lakshmikantham (2008). Then, this method is applied to study the stability for many classes of FrDEs such as: impulsive FrDE (see Agarwal (2016)), functional differential equations of fractional order (see Stamova (2016)), ... (see also Li's survey article (2011)). Stability conditions for linear FrDE expressed by fractional Lyapunov exponent and the linearized stability for semi-linear FrDE were studied by Cong (2014, 2016). In addition, using some other tools such as Gronwall inequality, comparison principle or Mittag-Leffler matrix function, several results on finite-time stability have obtained in the works of Kinh (2016), Lazarevic (2009), Li (2017) and Zhang (2016).

Regarding infinite dimensional FrDEs, stability results have been less known. In fact, using Lyapunov functional method is not impractical because the phase spaces are infinite dimensions and it is difficult to calculus the fractional derivative of Lyapunov function. Therefore, to study the stability for FrDE in infinite dimensional space, we need to develop new approaches.

Recently, the stability analysis in the sense of Lyapunov for a class of semilinear diffusion-wave equations involving impulsive effects and finite delays by using the fixed point approach. Anh and Ke (2015) established the existence of decay integral solutions for a class of neutral fractional differential equations in Banach spaces with unbounded delays by using the fixed point theory for condensing maps. Some other results on solvability, asymptotic stability, and existence of decay solutions for FrDE in infinite dimensional space can be referred to the works of Anh (2014), Ke (2014), Wang (2012) and Zhou (2010).

Finite-time dynamics has been studied extensively in the last two decades. The first motivation for the study is that, in analyzing the dynamics governed by the differential equation

$$\dot{x}(t) = f\big(x(t)\big), \tag{3}$$

one just gets the computed vector field in a bounded time-interval, i.e. $t \in [t_0, t_1]$. Even in the case (3) is given on the half-line, one may be interested in transient behavior of solutions, that is, the behavior on a compact interval $[t_0, t_1]$. The motivation for this study is that some model related to transport problems in fluid, biochemical networks or signal transduction (see Berger (2008), Rateitschak (2010)), in which processes happen in short time. Therefore the study of transient behavior plays a fundamental role and is physically more relevant than asymptotic behavior as $t \to \infty$. In this thesis, we use a concept of finite time dynamical for nonautonomous system in finite dimensional spaces proposed by Giesl (2012) to analyze the behavior of solutions at terminal time. A solution y of (3) is said to be attractive on $[0, T]$ if there exists $\eta > 0$ such that for every solution $x(\cdot, \xi)$ of (3) with initial datum ξ, it holds that

$$\|x(T,\xi) - y(T, y(0))\| < \|\xi - y(0)\|, \ \forall \xi \in B_\eta(y(0)) \backslash \{y(0)\},$$

here $B_\eta(y_0)$ stands for the ball centered at y_0 with radius η. If one has

$$\limsup_{\eta \searrow 0} \frac{1}{\eta} \sup_{\xi \in B_\eta(y(0))} \|x(T,\xi) - y(T, y(0))\| < 1,$$

then the solution y is called exponentially attractive on $[0, T]$. We refer to Berger (2011, 2008), Giesl (2012) and Haller (1998) for recent studies related to finite-time attractivity for ordinary differential equations. As far as we know, the finite-time attractivity analysis has not been made for FrDE in infinite dimensional space, especially for the class of equations which are reduced to the subdiffusion equations. Therefore we study the finite time attractivity for semilinear subdiffusion equations

$$\frac{d}{dt}\big(g_{1-\alpha} * [u - u(0)]\big)(t) = Au(t) + f\big(u(t)\big), \tag{4}$$

on the bounded time interval $[0, T]$, where the state function $x(\cdot)$ takes values in a Banach space X, A is a closed linear operator on X which generates a strongly continuous semigroup, $f : X \to X$ is a nonlinear function. Here $g_{1-\alpha} * v$, for $v \in L^1_{loc}(\mathbb{R}^+; X)$, denotes the Laplace convolution.

NDEs like (4) with A standing for second order elliptic partial differential operators have been used in mathematical physics to model dynamic processes in materials with memory. As mentioned in Kemppainen (2016), by replacing $g_{1-\alpha}$ with another locally integrable kernel, one can use the linear part of (4) to express many processes involving subdiffusion, superdiffusion and ultraslow-diffusion.

Using the Gronwall type inequality and local estimates of solutions (estimates with small initial data), we obtain the results on finite-time attractivity of the trivial solution (the zero solution) and of the nonzero solutions for equation (4).

In fact, when we establish a mathematical model for a real life problem by using an evolution system, there are usually two cases. In the first case, we can determine the initial datum and coefficients appearing in the system. Then we can solve or study qualitative properties of solutions by analysis methods. We call them forward problems. In the second case, we can not find every coefficients or are unable to measure the initial data. Then we have to determine not given coefficients or the initial values by added measurements. In this case, we face to a problem named inverse one. It is worth to emphasize that the inverse problems are ill-posed in Hadamard sense. Theses problems are more complicated, they require suitable approach in each concrete case. Therefore, the methods applied to inverse problems are very diverse.

The inverse problems for fractional derivative equations have been attracted attentions of mathematician in decades. There has been a number of results devoted to linear problems, where the source terms are determined by using the technique of eigenfunction expansion (see e.g., Kirane (2013), Sakamoto (2011), Wei (2013) and Zhang (2011)), the regularization method (Ruan (2017), Wei (2016)), the analytic Fredholm theorem (Tatar (2015)), or the unique continuation method (Jiang (2017)). In addition, the reader is referred to Jin (2015) and Kaltenbacher (2019) for various type of inverse problems for time-fractional diffusion equations, where the parameter to be identified can be initial data, sources, or other coefficients in the dynamic equation.

In comparison with linear cases, inverse problems of nonlinear type are much more involved and related results have been less known. In Luchko (2013), the problem of determining nonlinear source term was carried out by the aid of the maximum principle for fractional diffusion equations. Slodicka and Siskova (2016) employed the discretization method to deal with a problem of identifying the time-dependent source in a semilinear fractional diffusion equation. In Tatar and Ulusoy (2017), to determine the nonlinear diffusion term in a fractional diffusion equation, the optimization method was utilized. Dealing with an inverse source problem governed by a semilinear fractional wave equation in Wu (2014), the fixed point approach was chosen to prove a local existence result. Dwelling on this result, it is worth noting that, the local existence can not be extended to global one by continuation arguments as in the case of integer-order differential equations, due to the fact that solutions of fractional differential equations do not possess the semigroup property. We refer to Lorenzi (2012), (2014) for more results on inverse source problems that were done by using the fixed point technique.

In this thesis, we consider the problem of identifying parameter in a class of fractional differential variational inequalities (**FrIP**): For $\xi, \psi \in X$, find (x, u, z) satisfying the fractional differential variational inequality

$$D_0^\alpha x(t) = Ax(t) + B(u(t))z + h(x(t)), t \in (0, T], \tag{5}$$

$$\langle F(x(t)) + G(u(t)), v - u(t)\rangle \geq 0, \forall v \in \mathcal{K}, t \in [0, T], \tag{6}$$

$$x(0) = \xi, \tag{7}$$

and the condition

$$\int_0^T \varphi(s)x(s)ds = \psi, \tag{8}$$

where X be a Banach space, $\mathcal{U}$ a Hilbert space, $\mathcal{K}$ is a closed convex set in $\mathcal{U}$, (x, u) takes values in $X \times \mathcal{U}$, $z \in X$, $D_0^\alpha, \alpha \in (0, 1)$, is the fractional derivative in the Caputo sense. In this model, A is a closed linear operator on X, $\varphi \in C^1([0, T]; \mathbb{R})$ is a nonnegative nontrivial function, $B : \mathcal{U} \to \mathbb{R}$, $h : X \to X$, $F : X \to \mathcal{U}^*$ and $G : \mathcal{U} \to \mathcal{U}^*$ are given maps and the notation $\langle\cdot, \cdot\rangle$ stands for the canonical pairing between $\mathcal{U}$ and its dual $\mathcal{U}^*$.

Differential variational inequalities (DVIs) appear as a system which contains an evolution equation subject to a constraint formed by a variational inequality. DVIs were first systematically studied by Pang and Stewart (2008) as a general model for differential-algebraic equations, differential complementarity problems, etc. As a matter of fact, DVIs have been chosen to depict models, where dynamics and optimization intersect.

Let us mention some special cases of (5)-(6). Assume that $X = \mathbb{R}^n$, $\mathcal{K} = \mathcal{U} = \mathbb{R}^m$. Then our system has the following form

$$D_0^\alpha x(t) = \hat{F}(x(t), u(t), z),\ t \in (0, T],$$
$$\hat{G}(x(t), u(t)) = 0,\ t \in [0, T],$$

where $\hat{F}(x(t), u(t), z) = Ax(t) + B(u(t))z + h(x(t))$ and $\hat{G}(x(t), u(t)) = F(x(t)) + G(u(t))$, which is a differential-algebraic equation. This has been used, e.g., in modeling electric circuits as described in Pang (2008).

Now suppose that $\Omega \subset \mathbb{R}^n$ is a bounded smooth domain, $X = \mathcal{U} = \mathcal{K} = L^2(\Omega)$, $A = \Delta$ is the Laplacian associated with homogeneous Dirichlet boundary condition, and $G = -\Delta$. Then (5)-(6) can be interpreted as

$$\partial_t^\alpha y = \Delta y + \hat{h}(y, u, z) \text{ in } \Omega \times (0, T],$$

$$-\Delta u + F(y) = 0 \text{ in } \Omega \times [0,T],$$
$$y = u = 0 \text{ on } \partial\Omega \times [0,T],$$

where ∂_t^α stands for the Caputo derivative in t of order α, $\hat{h}(y,u,z) = B(u)z + h(y)$. The latter is of parabolic-elliptic system type, that has been employed for study of the motion of bacteria by chemotaxis (see Jäger (1992)), and image inpainting (see Jin (2010)).

Fractional DVIs were first introduced in Loi (2016), where the solvability is proved by using topological methods. In Ke (2015), some qualitative properties for a class of fractional DVIs were established. It should be mentioned that the models of fractional DVIs in Loi (2016) and Ke (2015) are formulated in finite dimensional spaces. If the evolution equation in DVIs represents a partial differential equation (PDE), we get infinite dimensional DVIs, which are the subject of recent investigations in Anh (2017), Liu (2017), (2016). In the last mentioned works, some questions related to existence and asymptotic behavior of solutions to DVIs associated with the first-order evolution equations in Banach spaces were addressed.

In this thesis, we propose a question on identifying the constraint term of the fractional DVI (9)-(11). More specifically, in the expression $B(u(t))z$ of equation (9), the factor $B(u(t))$ stands for the magnitude of constraint, the factor $z \in X$ is interpreted as the direction of constraint and supposed to be unknown. This factor will be identified by using the measurement condition (12).It is worth noting that, this condition is regarded as the mean value of the state function taking over $[0,T]$ in the case $\varphi(t) = T^{-1}$. The problem (**FrIP**) will be solved as follows. Under the assumption that A is a sectorial operator, we show that any integral solution to (9)-(11) is classical. Then by using the Schauder fixed point theorem, we prove the global existence of a triple (x,u,z) for given datum (ξ,ψ). With an additional assumption that the Lipschitz constant of F is small, we show the uniqueness and stability of solution by pointing out that the map $(\xi,\psi) \mapsto (x,u,z)$ is locally Lipschitzian as a map from $X \times D(A)$ to $C([0,T];X) \times C([0,T];\mathcal{U}) \times X$.

Moreover, in this thesis we are interested in two other NDE classes that have been recently studied in fluid dynamics. The first NDE class is concerned with the Basset type equation of the following form

$$\frac{d}{dt}\big(k_0 u + k * [u - u(0)]\big)(t) + Au(t) = f\big(u(t)\big), t \in (0,T] \tag{9}$$
$$u(0) = u_0, \tag{10}$$

where the state function $u(\cdot)$ takes values in a separable Hilbert space H, A is a linear operator on H, $f : H \to H$ is a nonlinear function.

Nonlocal differential equations like (9) naturally appear in a number of contexts, particularly in the heat transfer processes in memory materials (see Clément (1981), Prüss (2012)) and in the homogenization of an one-phase flow model in a fissured porous medium (see Amaziane (2007), Hornung (1990)). In the case $k_0 = 0$ and $k = g_{1-\alpha}, \alpha \in (0,1)$, equation (9) is the subdiffusion equation(6). In the case $k_0 > 0$ and $k = g_{1/2}$, equation (9) becomes the nonlinear Basset equation (see Ashyralyev (2011)).

The Basset equation was proposed in the 1910s by British mathematician A.B. Basset as he studied particle motion in an uncompressed viscous liquid under the action of gravity. McKee (1983) used the numerical method to find the approximate solution of the Basset type equation. The well-posedness of the of linear Basset equation has been established recently in Ashyralyev (2011), Bazhlekova (2014) and Hornung (1990). As far as we know, so far the qualitative results for (9) are less known. In this thesis, our aim is to find the appropriate conditions for k and f to prove the existence of solutions and finite time attractivity for solutions of (9)-(10) in case $k_0 > 0$.

Using the approach developed in Ke (2020), we derive the concept of mild solutions for inhomogeneous problems and prove a new Gronwall type inequality for (9)-(10). Using the contraction mapping principle and local estimates of solutions, we establish the global solvability and the finite-time attractivity. Consequence of attractivity, we prove the existence of periodic/anti-periodic solution of (9), i.e., the solutions satisfying $u(0) = \pm u(T)$.

The second NDE class related to the Rayleigh-Stokes equation has the following form

$$\partial_t u - \Delta u - \partial_t(m * \Delta u) = f(t,u) \ \text{ in } \Omega, t > 0, \tag{11}$$

$$\mathcal{B}u = 0 \ \text{ on } \partial\Omega,\ t \geq 0, \tag{12}$$

$$u(\cdot,0) = \xi \ \text{ in } \Omega,, \tag{13}$$

where $\partial_t = \frac{\partial}{\partial t}$, $m \in L^1_{loc}(\mathbb{R}^+)$ is a nonnegative function, f is a nonlinear function and $\xi \in L^2(\Omega)$ is given, $\mathcal{B}$ is a boundary operator in one of the following forms

$$\mathcal{B}u = u \text{ or } \mathcal{B}u = \nu \cdot \nabla u + \eta u,\ \eta > 0,$$

with ν being the outward normal vector to $\partial\Omega$.

We first mention some special cases of (11). If m is a nonnegative constant then (11) is the classical reaction-diffusion equation with nonlinear sources. In the case $m(t) = m_0 g_{1-\alpha}(t) = \dfrac{m_0 t^{-\alpha}}{\Gamma(1-\alpha)}$, $m_0 > 0$, $\alpha \in (0,1)$, our equation reads

$$\partial_t u - (1 + m_0 \partial_t^\alpha)\Delta u = f(t,u),$$

which is the generalized Rayleigh-Stokes equation (see Bazhlekova (2015)), here ∂_t^α denotes the fractional derivative of order α in the sense of Riemann-Liouville. In addition, if m is a regular function, e.g. $m \in C^1(\mathbb{R}^+)$, then (11) is a diffusion equation with memory, namely

$$\partial_t u - (1 + m(0))\Delta u - \int_0^t m'(t-s)\Delta u(s)ds = f(t,u),$$

which has been a topic of an extensive study, see e.g., Cannarsa (2013), Conti (2014), Miller (1978).

The generalized Rayleigh-Stokes equation is used to describe behavior of non-Newtonian fluids in materials possessing both elasticity and viscosity. Accordingly, the generalized Rayleigh-Stokes equation is employed to study practical applications in industry and engineering (see Bazhlekova (2015)). As a matter of fact, there have been lots of works devoted to finding numerical methods for solving Rayleigh-Stokes problem, see e.g., Bazhlekova (2015), Chen (2013).

Recently, the final value problem for the generalized Rayleigh-Stokes equation is discussed in the works of Luc (2019) and Tuan (2019). In this thesis, we study the solvability and stability for problem (11)-(13) in the case the kernel function m being possibly singular (for example, m is unbounded in the neighborhood of $t = 0$).

To this end, we assume that $m \in L^1_{loc}(\mathbb{R}^+)$ such that $a_\gamma(t) := 1 + \gamma m(t)$ is completely positive for any $\gamma > 0$. Based on this assumption, we contruct a representation of mild solutions and prove a new Gronwall type inequality for (11)-(13). Using this and local estimates of solutions, we obtain the global solvability and stability results. In addition, when the uniqueness is not guaranteed, we prove that there exists a compact set of decay solutions for (11) - (13), by applying the fixed point theorem for condensing maps.

2. Purpose, objects and scope of the thesis

2.1. Purpose: The thesis focus on studying some qualitative properties for some classes of NDEs, including: the finite-time attractivity, the stability of solutions; the solvability, the uniqueness and Lipschitz stability for the problem of identifying parameter.

2.2. Objects: In the thesis, we consider some following types of NDEs

- ⋆ The first type: Subdiffusion equations.
- ⋆ The second one: Basset type equations.
- ⋆ The third one: Rayleigh-Stokes type equations.
- ⋆ The fourth one: Fractional differential variational inequalities.

2.3. Scope: The scope of the thesis is defined by the following contents

⋆ **Content 1:** Study the finite-time attractivity of solutions for two classes of NDEs: Subdiffusion equations and Basset type equations;

⋆ **Content 2:** Study the asymptotic stability of solutions for semilinear Rayleigh-Stokes type evolutionary equations;

⋆ **Content 3:** Study the solvability, the uniqueness and the stability for the problem of identifying parameter in a class of fractional differential variational inequalities.

3. Research Methods

In this thesis, we employ theory of operator, theory of stability and fixed point arguments. In addition, for concrete content, we use suitable techniques.

- To prove the existence of solutions, decay solutions we make use of MNCs estimates and fixed point theory for condensing maps.
- To prove the finite-time attractivity, we use Gronwall type inequalities and prior estimates techniques.

- To prove the asymptotic stability of solutions, we use Gronwall type inequalities and prior estimates techniques.
- To study the solvability, the uniqueness and stability for the problem of identifying parameter in a class of fractional differential variational inequalities, the regularity for fractional diffusion equations and fixed point techniques are employed.

4. Results of thesis

The thesis achieved the following main results:

1. Proving the existence of integral solutions and the finite-time attractivity of the zero solution as well as the nonzero solutions for two classes of semilinear NDEs: subdiffusion equations and Basset type equations, where the nonlinear term has superlinear growth.
2. Proving the existence and the asymptotic stability of solutions for a class of semilinear Rayleigh-Stokes type evolutionary equations. Especially, in the case of nonuniqueness, we prove that there exists a nonempty compact set of decay solutions.
3. Proving the solvability, the uniqueness and stability for the problem of identifying parameter in a class of fractional differential variational inequalities.

6. Structures of thesis

Together with the Introduction, Conclusion, Authors works related to the thesis that have been published and References, the thesis includes four chapters: Chapter 1 is devoted to present some preliminaries. In Chapter 2, we present the finite-time attractivity for two classes of nonlocal differential equation: subdiffusion equation and Basset type equation. Chapter 3 is devoted to studying the asymptotic stability and the existence of decay solutions for a class of semilinear Rayleigh-Stokes type equations. Chapter 4 presents problem of identifying parameter in a class of fractional differential variational inequalities.

Chapter 1

PRELIMINARIES

In this chapter, we present some preliminaries including: some results about fractional caculus; resolvent theory; measure of noncompactness; fixed point principles; scalar Volterra equations with completely positive kernels.

1.1. FUNCTIONAL SPACES

In this section, we recall some important functional spaces which are used in next chapters.

1.2. Fractional calculus

In this section, we recall some concept and properties related to fractional derivatives and fractional integral.

1.3. Laplace transform

In this section we recall the definition and some basic properties of the Laplace transform, the inverse Laplace transform.

1.4. Measure of noncompactness (MNC) and MNS estimates

In this section, we present concept of MNC and some estimates related to Hausdorff MNC.

1.5. Condensing map and some fixed point theorems

In this section, we recall some fixed point principles for condensing map.

1.6. Semigroup theory

This section is devoted to present some definitions and results in theory of semigroup.

1.7. The Cauchy problem for fractional differential equations

This section gives a representation of integral solution to the Cauchy problem for the subdiffusion equations.

1.8. Volterra integral equations

In this section, we recall some of the results for the solutions of the Volterra integral equations with completely positive kernels.

Chapter 2

FINITE-TIME ATTRACTIVITY FOR SOME CLASSES OF SEMILINEAR NONLOCAL EVOLUTION EQUATIONS

We prove the existence and finite-time attractivity of solutions to two classes of semilinear nonlocal evolution equations: subdiffusion equations and Basset type equations. Our analysis is based on the resolvent theory, the fixed point theory for condensing maps and the local estimates of solutions.

The content of this chapter is written based on the papers [1] and [2] in the author's works related to the thesis that has been published.

2.1. Finite-time attractivity for semilinear subdiffusion equations

In this section, we study the solvability and the finite-time attractivity of solutions for semilinear subdiffusion equations.

2.1.1. Problem setting

Let $(X, \|\cdot\|)$ be a Banach space and $\alpha \in (0,1)$. Consider the following problem

$$\frac{d}{dt}\big(g_{1-\alpha} * [u - u(0)]\big)(t) = Au(t) + f\big(u(t)\big), t \in [0,T], \tag{2.1}$$

where $g_{1-\alpha}(t) = t^{-\alpha}/\Gamma(1-\alpha), \alpha \in (0,1), t > 0$, the state $u(\cdot)$ takes values in a Banach space X, A is a closed linear operator on X which generates a strongly continuous semigroup, $f : X \to X$ is a nonlinear function.

2.1.2. Existence results

To study the solvability of equation (2.1), we assume

(**HA**) The C_0-semigroup $\{S(t)\}_{t\geq 0}$ generated by A is norm-continuous and globally bounded, i.e. there is $M \geq 1$ such that

$$\|S(t)u\| \leq M\|u\|, \forall t \geq 0, \forall u \in X.$$

(**HF**) The nonlinear function $f : X \to X$ is continuous and satisfies:

(1) the growth condition

$$\|f(u)\| \leq \Psi(\|u\|), \forall u \in X,$$

where $\Psi : \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous and nondecreasing function;

(2) if $S(\cdot)$ is non-compact then for any bounded set $\Omega \subset X$, we have

$$\chi\big(f(\Omega)\big) \leq k\, \chi(\Omega), k \in \mathbb{R}^+.$$

Given $\xi \in X$, we define *the solution operator* $\Sigma : C([0,T];X) \to C([0,T];X)$ as follows

$$\Sigma(u)(t) = \mathcal{S}_\alpha(t)\xi + \int_0^t (t-s)^{\alpha-1}\mathcal{P}_\alpha(t-s)f(u(s))\,ds.$$

It is easy to verify that y is an integral solution of (2.1) iff it is a fixed point of the solution operator Σ. The next lemma will be used to show the condensivity of Σ.

Lemma 2.1. *Let the hypotheses* (**HA**) *and* (**HF**) *hold. Then*

$$\chi^*(\Sigma(\Omega)) \le \Big(\sup_{t\in[0,T]} 4k\int_0^t (t-s)^{\alpha-1}\|\mathcal{P}_\alpha(t-s)\|_\chi ds\Big)\chi^*(\Omega).$$

The following theorem states the result of solvability for our problem. (2.1).

Theorem 2.1. *Let the hypotheses* (**HA**) *and* (**HF**) *hold. If there exists* $R > 0$ *such that*

$$\frac{\Gamma(\alpha+1)R}{\Gamma(\alpha+1)\|\xi\| + T^\alpha\Psi(R)} \ge M, \tag{2.2}$$

then the solution set to (3) *is nonempty and compact.*

2.1.3. Finite-time attractivity

In this section, we prove some results on finite-time attractivity of solutions to (2.1). By $\mathbb{S}(\xi)$, we denote the solution set of (2.1) with respect to the initial datum ξ. We adopt the following concept of finite-time attractivity of solutions to (2.1).

Definition 2.1 (Finite-time attractivity)**.** Let $y \in C([0,T];X)$ be a solution of (2.1). Then y is exponentially attractive on $[0,T]$, provided that

(i) y is called attractive on $[0,T]$ if there exists an $\eta > 0$ such that

$$\|u(T,\xi) - y(T,y(0))\| < \|\xi - y(0)\|,$$

for all $\xi \in B_\eta(y(0))\backslash\{y(0)\}$ and $u \in \mathbb{S}(\xi)$.

(ii) y is called exponentially attractive on $[0,T]$ if

$$\limsup_{\eta\searrow 0}\frac{1}{\eta}\sup_{\xi\in B_\eta(y(0))}\sup_{u\in\mathbb{S}(\xi)}\|u(T,\xi) - y(T,y(0))\| < 1.$$

One can easily verify from the definition that exponential attractivity implies attractivity. The following lemma gives a sufficient condition for exponential attractivity.

Lemma 2.2. *Let* $y \in C([0,T];X)$ *be a solution of* (2.1)*. Then* y *is exponentially attractive on* $[0,T]$*, provided that*

$$\limsup_{\|\xi\|\to 0}\sup_{u\in\mathbb{S}(y(0)+\xi)}\frac{\|u(T,y(0)+\xi) - y(T,y(0))\|}{\|\xi\|} < 1. \tag{2.3}$$

In order to analyze the finite-time attractivity of solutions of (3), we replace the hypotheses (**HA**) and (**HF**) by the following ones.

($\mathbf{A}^*$) The semigroup $S(\cdot)$ generated by A is norm-continuous and there exist $M \geq 1, \beta > 0$ such that

$$\|S(t)u\| \leq Me^{-\beta t}\|u\|, \forall t \geq 0, \forall u \in X.$$

($\mathbf{F}^*$) The function f satisfies (**HF**) with Ψ being locally Lipschitz and $\Psi(0) = 0$. Furthermore, $\|f(v)\| = \gamma\|v\| + o(\|v\|)$ as $\|v\| \to 0$, for some $\gamma < \frac{\beta}{M}$.

Lemma 2.3. *Let* ($\mathbf{A}^*$) *and* ($\mathbf{F}^*$) *hold. Then*

$$\limsup_{\|\xi\|\to 0} \sup_{u\in\mathbb{S}(\xi)} \|u(t)\| = 0, \ \forall t \in (0, T].$$

Theorem 2.2. *Let* ($\mathbf{A}^*$) *and* ($\mathbf{F}^*$) *hold. Then the zero solution of* (2.1) *is exponentially attractive on* $[0, T]$ *provided that*

$$E_{\alpha,1}(-(\beta - \gamma M)T^\alpha) < \frac{1}{M}. \tag{2.4}$$

As a consequence, we are able to prove a linearized attractivity result for (2.1).

Corollary 2.1. *Let* (**HA**) *hold. Assume* $f \in C^1(X)$ *such that*

(1) $f(0) = 0$;

(2) f *obeys* (**F**) *with* Ψ *being locally Lipschitz;*

(3) $A_0 = A + Df(0)$ *is the generator of an exponentially stable semigroup* $\{S_0(t)\}_{t\geq 0}$, *that is*

$$\|S_0(t)\|_{op} \leq e^{-\beta t}, \ \forall t \geq 0,$$

for some $\beta > 0$.

Then the zero solution to (2.1) *is attractive on* $[0, T]$.

In order to prove the attractivity of nonzero solutions, we replace ($\mathbf{F}^*$) by the following hypothesis.

($\mathbf{F}^\sharp$) The function f satisfies (**F**)(2) and the following estimate

$$\|f(u) - f(v)\| \leq \Psi(\|u - v\|), \ \forall u, v \in X,$$

where $\Psi \in C(\mathbb{R}^+; \mathbb{R}^+)$ is a nondecreasing and locally Lipschitz function satisfying $\Psi(r) = \gamma r + o(r)$ as $r \to 0, \gamma < \frac{\beta}{M}$.

Theorem 2.3. *Let* ($\mathbf{A}^*$) *and* ($\mathbf{F}^\sharp$) *hold. If condition* (2.4) *is testified then every solution to* (3) *is exponentially attractive on* $[0, T]$.

2.1.4. Application

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with smooth boundary $\partial\Omega$. Consider the following fractional partial differential equation

$$\partial_t^\alpha u(x, t) = \Delta_x u(x, t) + \tilde{f}(u(x, t)), \ \alpha \in (0, 1), \ t \in [0, T], \tag{2.5}$$

$$u = 0 \text{ on } \partial\Omega, \tag{2.6}$$

$$u(x,0) = \xi(x),\ x \in \Omega. \tag{2.7}$$

In the above model, ∂_t^α stands for the Caputo fractional derivative of order α with respect to t, Δ_x denotes the Laplace operator with respect to x, and $\tilde{f} : \mathbb{R} \to \mathbb{R}$ is a continuous function.

Let $X = C_0(\overline{\Omega}) = \{v \in C(\overline{\Omega}) : v = 0 \text{ on } \partial\Omega\}$, endowed with the norm $\|v\| = \sup_{x\in\overline{\Omega}} |v(x)|$. Put $A = \Delta$ with the domain $D(A) = \{v \in C_0(\overline{\Omega}) \cap H_0^1(\Omega) : \Delta v \in C_0(\overline{\Omega})\}$, and define $f : C_0(\overline{\Omega}) \to C_0(\overline{\Omega})$ as follows

$$f(v)(x) = \tilde{f}(v(x)),\ \forall v \in C_0(\overline{\Omega}).$$

Then (2.5)-(2.7) is in the fashion of (2.1). It is known that A generates a contraction C_0-semigroup $\{S(t)\}_{t\geq 0}$, compact on X. Hence, (**A**) is fulfilled.

It should be noted that, $S(\cdot)$ satisfies

$$\|S(t)\|_{op} \leq Me^{-\lambda_1 t},\ M = \exp\left(\frac{\lambda_1|\Omega|^{2/N}}{4\pi}\right),$$

where λ_1 is the first eigenvalue of $(-\Delta)$ in $H_0^1(\Omega)$, that is

$$\lambda_1 = \sup\left\{\frac{\int_\Omega |\nabla u|^2 dx}{\int_\Omega u^2 dx} : u \in H_0^1(\Omega), u \neq 0\right\},$$

and $|\Omega|$ is the volume of Ω. Therefore (**A***) is satisfied.

We first consider the case $\tilde{f}$ is superlinear, that is

$$|\tilde{f}(z)| \leq k\,|z|^p, \forall z \in \mathbb{R},\ \text{for some } k > 0,\ p > 1.$$

Then $f(0) = 0$ and $\|f(v)\| \leq k\|v\|^p$ for all $v \in C_0(\overline{\Omega})$. It is easily seen that f satisfies (**F***) with $\gamma = 0$. Due to Theorem 2.2, the zero solution of (2.5) is exponentially attractive on $[0,T]$ provided that

$$\exp\left(\frac{\lambda_1|\Omega|^{2/N}}{4\pi}\right) E_{\alpha,1}(-\lambda_1 T^\alpha) < 1.$$

In fact, the last condition is imposed on T, which requires that $T > T^*$ with a $T^* = T^*(\Omega, N) > 0$. We are now going to relax this requirement by assuming that $\tilde{f} \in C^2(\mathbb{R})$ such that $\tilde{f}'(0) < 0$. Since $\tilde{f}$ is twice differentiable, one has $f \in C^1(C_0(\overline{\Omega}))$. Noting that $Df(0) = \tilde{f}'(0)I$, the semigroup $S_0(\cdot)$ generated by $A_0 = A + Df(0)$ is given by

$$S_0(t) = e^{\tilde{f}'(0)t} S(t),\ t \geq 0,$$

and we have

$$\|S_0(t)\|_{op} \leq e^{\tilde{f}'(0)t},\ t \geq 0,$$

due to the fact that $S(\cdot)$ is a contraction semigroup. Now employing Corollary 2.1, we can state that the zero solution of (2.5) is exponentially attractive on $[0,T]$ for any $T > 0$.

2.2. Finite-time attractivity for semilinear Basset type evolutionary equations

In this section, we study the solvability and the finite-time attractivity of solutions for semilinear Basset type evolution equations.

2.2.1. Problem setting

Let H be a separable Hilbert space and $T > 0$. Consider the following problem

$$\frac{d}{dt}\big(k_0 u + k * [u - u(0)]\big)(t) + Au(t) = f\big(u(t)\big), t \in (0, T] \tag{2.8}$$

$$u(0) = u_0, \tag{2.9}$$

where the state function $u(\cdot)$ takes values in H, $k_0 > 0$, $k \in L^1_{loc}(\mathbb{R}^+)$, A is a linear operator on H, $f : H \to H$ is a nonlinear function.

In order to examine (2.8)-(2.9), we make the following standing assumptions.

(Hk) $k_0 > 0$ and the kernel $k \in L^1_{loc}(\mathbb{R}^+)$ is a nonnegative and nonincreasing function.

(Ha) $A : D(A) \to H$ is densely defined, self-adjoint and positively definite operator with a compact resolvent.

2.2.2. Formula of solution for the linear problem

Assumption that (Hk) hold. Consider the linear problem

$$\frac{d}{dt}\big(k_0 u + k * [u - u_0]\big)(t) + Au(t) = f(t), t \in (0, T] \tag{2.10}$$

$$u(0) = u_0, \tag{2.11}$$

where $f \in C([0, T], H)$. Denote ℓ is the unique solution of the integral equation

$$k_0 \ell + k * \ell = 1 \text{ on } \mathbb{R}^+. \tag{2.12}$$

By the assumption (Ha), there exists a nondecreasing sequence $\{\lambda_n\}_{n=1}^{\infty}$,

$$0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \leq \cdots, \lambda_n \to +\infty \text{ khi } n \to +\infty,$$

and a system of vectors $\{e_n\}_{n=1}^{\infty} \subset D(A)$ such that $\{e_n\}_{n=1}^{\infty}$ which forms an orthonormal basis of H and $Ae_n = \lambda_n e_n$, for all$n \in \mathbb{N}$. We note that equation (2.10) is equivalent to the Volterra equation

$$u + \ell * Au = u_0 + \ell * f.$$

Let $u(t) = \sum_{n=1}^{\infty} u_n(t)e_n$, $u_0 = \sum_{n=1}^{\infty} u_{0,n}e_n$, $f(t) = \sum_{n=1}^{\infty} f_n(t)e_n$. Therefore, we find that $u_n(t) = s_{\lambda_n}(t)u_{0,n} + (r_{\lambda_n} * f)(t)$, where $s_{\lambda_n}, r_{\lambda_n}, n = 1, 2, \ldots$ are unique solutions of the following Volterra integral equations

$$s_\mu(t) + \lambda_n\big(\ell * s_\mu\big)(t) = 1, \quad t \geq 0, \tag{2.13}$$

and

$$r_\mu(t) + \lambda_n\big(\ell * r_\mu\big)(t) = \ell(t), \quad t \geq 0. \tag{2.14}$$

So $u(t) = S(t)u_0 + \int_0^t R(t - s)f(s)ds$, where $\{S(t)\}_{t\geq 0}$ and $\{R(t)\}_{t>0}$ defined by

$$S(t)z = \sum_{n=1}^{\infty} s_{\lambda_n}(t)z_n e_n, t \geq 0, R(t)z = \sum_{n=1}^{\infty} r_{\lambda_n}(t)z_n e_n, t > 0, z \in H, \text{ and}$$

$$\int_0^t R(t - s)f(s)ds = \sum_{n=1}^{\infty}\big(r_{\lambda_n} * f_n\big)(t)e_n.$$

2.2.3. Existence results

Definition 2.2. A function $u \in C([0,T];H)$ is called a mild solution of problem (2.8)-(2.9) on the interval $[0,T]$ iff

$$u(t) = S(t)u_0 + \int_0^t R(t-s)f\big(u(s)\big)\,ds, \text{ for all } t \in [0,T].$$

As far as the nonlinearity f is concerned, we assume that

(Hf) The nonlinear function $f : H \to H$ is locally Lipschitz, that is

$$\|f(u_1) - f(u_2)\| \le L(r)\|u_1 - u_2\|, \forall u_1, u_2 \in B_r,$$

where B_r is the closed ball centered at origin with radius r in $C([0,T];H)$ and $\limsup_{r\to 0} L(r) < \lambda_1$.

We analyze the first case for global solvability in the following theorem.

Theorem 2.4. *Let the hypotheses* $(Hk), (Ha)$ *and* (Hf) *hold. If* $f(0) = 0$*, then there exist* $r > 0$ *and* $\delta > 0$ *such that the problem* (2.8)-(2.9) *admits a unique global mild solution* $u \in B_r$ *provided that* $\|u_0\| \le \delta$.

In the special case the nonlinear function f is globally Lipschitz, we also obtain the global solvability of the problem (2.8)-(2.9) without any restriction on the initial data.

Theorem 2.5. *Let the hypotheses* $(Hk), (Ha)$ *and* (Hf) *hold with* $L(r)$ *being a constant. Then the problem* (2.8)-(2.9) *has a unique global mild solution.*

2.2.4. Finite-time attractivity

This section presents our main result on finite-time attractivity of solutions to (2.8).

Theorem 2.6. *Let the assumptions of Theorem 2.4 hold. Then there exists* $\delta > 0$ *such that, every solution* u *of* (2.8) *with* $\|u(0)\| \le \delta$ *is exponentially attractive on* $[0,T]$.

In the case the nonlinear function f is globally Lipschitzian, we get the following result.

Theorem 2.7. *If the assumptions of Theorem 2.5 hold, then every solution of* (2.8) *is exponentially attractive on* $[0,T]$*, provided that* $L < \lambda_1$.

By the same conditions ensuring the attractivity, we prove the solvability result of the following problem

$$\frac{d}{dt}\big(k_0 u + k * [u - u(0)]\big)(t) + Au(t) = f\big(u(t)\big), t \in (0,T] \tag{2.15}$$

$$u(0) = g(u), \tag{2.16}$$

where $g : C([0,T];H) \to H$ satisfies the following assumption.

(Hg) There exists $\tau \in (0,T]$ such that

$$\|g(u_1) - g(u_2)\| \le \sup_{s\in[\tau,T]} \|u_1(s) - u_2(s)\|, \forall u_1, u_2 \in C([0,T];H).$$

By mild solution to problem (2.15)-(2.16), we mean a function $u \in C([0,T];H)$ satisfying

$$u(t) = S(t)g(u) + \int_0^t R(t-s)f\big(u(s)\big)\, ds, \forall t \in [0,T].$$

We are now in a position to prove the solvability of problem (2.15)-(2.16).

Theorem 2.8. *Let the hypotheses of Theorem 2.5 and* (Hg) *hold. Then the problem* (2.15)-(2.16) *has a mild solution.*

2.2.5. Application

Let $\alpha \in (0,1)$. Consider the following nonlinear integrodifferential equation

$$\begin{aligned} \partial_t u(x,t) + \partial_t^\alpha u(x,t) &= \partial_x^2 u(x,t) \\ &\quad + h\Big(\int_0^1 u^2(x,t)dx\Big)u(x,t), x \in (0,1), t \in (0,T] \qquad (2.17) \end{aligned}$$

$$u(0,t) = u(1,t) = 0, t \geq 0, \qquad (2.18)$$

$$u(x,0) = \xi(x),\ x \in [0,1]. \qquad (2.19)$$

In the above model, ∂_t^α stands for the Caputo fractional derivative of order α, ∂_x denotes the generalized derivative in variable x. In this case the kernel k is given by $k(t) = g_{1-\alpha}(t)$.

Let $H = L^2(0,1)$. The inner product and the norm in H are given by

$$(u,v) = \int_0^1 u(x)v(x)dx, \qquad ||v|| = \Big(\int_0^1 |v(x)|^2 dx\Big)^{\frac{1}{2}}.$$

Let $A = -\partial_x^2$ with the domain $D(A) = H^2(0,1) \cap H_0^1(0,1)$. It is known that A is a densely defined, self-adjoint and positive operator with domain $D(A)$ and has a compact resolvent on H. Moreover, the eigenvalues of A consists of $\lambda_n = n^2\pi^2, n = 1,2,\ldots,$ with corresponding eigenvectors $e_n = \sqrt{2}\sin(nx), n \geq 1$, which form an orthonormal basis in H. So the hypothesis (Ha) is verified.

Let $f(v)(x) = h\Big(\int_0^1 v^2(x)\ dx\Big)v(x), v \in L^2(0,1)$. Clearly, the problem (2.5)-(2.7) is a model of (2.8)-(2.9) with $k_0 = 1$. A simple computation shows that $(-1)^n k^{(n)}(t) \geq 0, \forall n \in \mathbb{N}, t > 0$ and hence the kernel k is completely monotonic. Consequently, the hypothesis (Hk) is satisfied.

Regarding the nonlinearity in equation (2.5), we assume that the function h belongs to $C^1\big(\mathbb{R}\big)$ and $|h(r)| \leq a + b|r|^\beta$, for some nonnegative constants a, b, β. One can check that f maps $L^2(0,1)$ into itself and for all $v_1, v_2 \in L^2(0,1)$ such that $\|v_1\|, \|v_2\| \leq r$, the following holds

$$\|f(v_1) - f(v_2)\| \leq \big(2r^2 \sup_{z\in[0,r^2]} |h'(z)| + a + br^{2\beta}\big)\|v_1 - v_2\|.$$

Hence, the hypothesis (Hf) is fulfilled with $L(r) = 2r^2 \sup_{z\in[0,r^2]} |h'(z)| + a + br^{2\beta}$. Obviously, $\lim_{r\to 0} L(r) = a$. Therefore, if $a < \pi^2$, then every solution of (2.17)-(2.19) with ξ small enough is attractive on $[0,T]$.

Chapter 3

STABILITY FOR SEMILINEAR RAYLEIGH-STOKES TYPE EVOLUTION EQUATIONS

In this chapter, we analyze the solvability, the stability and the existence of decay solutions for semilinear Rayleigh-Stokes type evolution equations. Our analysis is based on the theory of completely positive functions, local estimates and fixed point arguments.

The content of this chapter is written based on the paper [4] in the author's works related to the thesis that has been published.

3.1. Problem setting

Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with smooth boundary $\partial\Omega$. Consider the following problem

$$\partial_t u - \Delta u - \partial_t(m * \Delta u) = f(t, u) \ \text{ in } \Omega, t > 0, \tag{3.1}$$

$$\mathcal{B}u = 0 \ \text{ on } \partial\Omega,\ t \geq 0, \tag{3.2}$$

$$u(\cdot, 0) = \xi \ \text{ in } \Omega, \tag{3.3}$$

where $m \in L^1_{loc}(\mathbb{R}^+)$ is a nonnegative function, f is a nonlinear function and $\xi \in L^2(\Omega)$ is given, $\mathcal{B}$ is a boundary operator in one of the following forms

$$\mathcal{B}u = u \text{ or } \mathcal{B}u = \nu \cdot \nabla u + \eta u,\ \eta > 0,$$

with ν being the outward normal vector to $\partial\Omega$.

3.2. Formula of solution for the linear problem

In this section, we find a representation for solution of the linear problem

$$\partial_t u - \Delta u - \partial_t(m * \Delta u) = F \ \text{ in } \Omega, t \in (0, T], \tag{3.4}$$

$$\mathcal{B}u = 0 \ \text{ on } \partial\Omega,\ t \in [0, T], \tag{3.5}$$

$$u(\cdot, 0) = \xi \ \text{ in } \Omega, \tag{3.6}$$

where $F \in C([0, T]; L^2(\Omega))$. Let $\{\varphi_n\}_{n=1}^\infty$ be an orthonormal basis of $L^2(\Omega)$ consisting of eigenfunctions of $-\Delta$ subject to the homogeneous boundary condition, i.e.,

$$-\Delta\varphi_n = \lambda_n \varphi_n \text{ in } \Omega,\ \mathcal{B}\varphi_n = 0 \text{ on } \partial\Omega,$$

where one can assume that $0 < \lambda_1 \leq \lambda_2 \leq ...,\ \lambda_n \to \infty$ as $n \to \infty$. For $\beta \in \mathbb{R}$, the fractional power operator $(-\Delta)^\beta$ is defined as follows

$$(-\Delta)^\beta v = \sum_{n=1}^\infty \lambda_n^\beta (v, e_n) e_n,$$

$$D((-\Delta)^\beta) = \{v \in L^2(\Omega) : \sum_{n=1}^\infty \lambda_n^{2\beta} (v, e_n)^2 < \infty\},$$

here the notation $(\cdot,\cdot)$ denotes the inner product in $L^2(\Omega)$. We assume that $m \in L^1_{loc}(\mathbb{R}^+)$ satisfies the standing assumption:

(M) *The function $m \in L^1_{loc}(\mathbb{R}^+)$ is nonnegative such that the function $a_\gamma(t) := 1 + \gamma m(t)$ is completely positive for any $\gamma > 0$.*

Using the Fourier series method, the solution of problem (3.4)-(3.6) has the following representation

$$u(\cdot,t) = S(t)\xi + \int_0^t S(t-\tau)F(\cdot,\tau)d\tau, \tag{3.7}$$

where $S(t)$ is *the resolvent operator* defined by

$$S(t)\xi = \sum_{n=1}^{\infty} \omega(t,\lambda_n)\xi_n\varphi_n,\ \xi \in L^2(\Omega), \tag{3.8}$$

here $\omega(t,\lambda_n), n = 1, 2, \dots$ be the solution of the following problem

$$\omega'(t) + \lambda_n\omega(t) + \lambda_n(m * \omega)'(t) = 0 \text{ for } t > 0,\ \omega(0) = 1. \tag{3.9}$$

3.3. Solvability and stability

Based on representation (3.7), we give the following definition of mild solution for (3.1)-(3.3).

Definition 3.1. A function $u \in C([0,T]; L^2(\Omega))$ is said to be a mild solution to the problem (3.1)-(3.3) on $[0,T]$ iff

$$u(\cdot,t) = S(t)\xi + \int_0^t S(t-\tau)f(\tau, u(\cdot,\tau))d\tau \text{ for any } t \in [0,T].$$

We now prove a global solvability result for (3.1)-(3.3).

Theorem 3.1. *Let* (**M**) *hold. Assume that the nonlinearity function $f : [0,T] \times L^2(\Omega) \to L^2(\Omega)$ satisfies*

(F1) *f is continuous such that $f(\cdot,0) = 0$ and $\|f(t,v_1) - f(t,v_2)\| \le \kappa(r)\|v_1 - v_2\|$ for all $v_1, v_2 \in B_r$, $t \in [0,T]$, where B_r is the closed ball in $L^2(\Omega)$ centered at origin with radius r, $\kappa(\cdot)$ is a nonnegative function such that $\limsup_{r\to 0} \kappa(r) = \ell \in [0,\lambda_1)$.*

Then there exists $\delta > 0$ such that the problem (3.1)-(3.3) *has a unique mild solution on* $[0,T]$, *provided $\|\xi\| \le \delta$.*

In the next theorem, we will show that if the nonlinear function f has a sublinear growth, the smallness condition on initial data can be relaxed.

Theorem 3.2. *Let* (**M**) *hold with m being nonincreasing. Assume that f satisfies the condition*

(F2) *f is continuous such that $\|f(t,v)\| \le p(t)\|v\| + q(t)$, for all $v \in L^2(\Omega)$, where $p, q \in L^1(0,T)$ are nonnegative functions.*

Then the problem (3.1)-(3.3) *has at least one mild solution on* $[0, T]$.

We are in a position to formulate some results on stability for our system.

Theorem 3.3. *Let the assumption of Theorem 3.2 hold for any* $T > 0$ *with* $p \in L^\infty(\mathbb{R}^+)$ *and* $q \in L^1_{loc}(\mathbb{R}^+)$ *such that* $\|p\|_\infty < \lambda_1$ *and* $\omega * q$ *is a bounded function. Then there exists an absorbing set for solution of* (3.1)-(3.3) *with arbitrary initial data. Moreover, if* $q = 0$*, then the zero solution of* (3.1) *is asymptotically stable.*

Theorem 3.4. *Let the hypotheses of Theorem 3.1 hold for any* $T > 0$*. Then the zero solution of* (3.1) *is asymptotically stable.*

Considering the case when f is globally Lipschitzian, we have a stronger result.

Theorem 3.5. *Let* (**M**) *hold. If* f *satisfies the Lipschitz condition as follows*

$$\|f(t, v_1) - f(t, v_2)\| \le \kappa_0 \|v_1 - v_2\|, \text{ for} t \in \mathbb{R}^+, v_1, v_2 \in L^2(\Omega),$$

where $\kappa_0 \in [0, \lambda_1)$*, then every mild solution of* (3.1)-(3.2) *is asymptotically stable.*

3.4. Existence of decay solutions

In this section, we consider the problem (3.1)-(3.3) under the assumption that, the nonlinearity is non-Lipschitzian and possibly superlinear. More precisely,

(**F**3) $f : \mathbb{R}^+ \times L^2(\Omega)) \to L^2(\Omega)$ is a continuous mapping such that

$$\|f(t, v)\| \le p(t)G(\|v\|), \ \forall t \in \mathbb{R}^+, v \in L^2(\Omega),$$

where $p \in L^1_{loc}(\mathbb{R}^+)$ is a nonnegative function and $G \in C(\mathbb{R}^+)$ is a nonnegative and nondecreasing function such that

$$\limsup_{r \to 0} \frac{G(r)}{r} \cdot \sup_{t \ge 0} \int_0^t \omega(t - \tau, \lambda_1) p(\tau) d\tau < 1, \tag{3.10}$$

and

$$\lim_{T \to \infty} \sup_{t \ge T} \int_0^{\frac{t}{2}} \omega(t - \tau, \lambda_1) p(\tau) d\tau = 0. \tag{3.11}$$

The following theorem gives the main result of this section.

Theorem 3.6. *Let* (**M**) *and* (**F**3) *hold. Then there exists* $\delta > 0$ *such that, the problem* (3.1)-(3.3) *has a compact set of decay solutions, provided* $\|\xi\| \le \delta$.

Chapter 4

IDENTIFYING PARAMETER IN A CLASS OF FRACTIONAL DIFFERENTIAL VARIATIONAL INEQUALITIES

In this chapter, we study the solvability and the stability for the problem of identifying parameter in a class of fractional differential variational inequalities. Our approach is based on a regularity analysis for fractional diffusion equations and fixed point techniques.

The content of this chapter is written based on the paper [3] in the author's works related to the thesis that has been published.

4.1. Problem setting

Let X be a Banach space and $\mathcal{U}$ a Hilbert space. Given closed convex set $\mathcal{K} \subset \mathcal{U}$, we consider the identification problem: (**FrIP**) For $\xi, \psi \in X$, find (x, u, z) satisfying the fractional differential variational inequality

$$D_0^\alpha x(t) = Ax(t) + B(u(t))z + h(x(t)), t \in (0, T], \tag{4.1}$$

$$\langle F(x(t)) + G(u(t)), v - u(t)\rangle \geq 0, \forall v \in \mathcal{K}, t \in [0, T], \tag{4.2}$$

$$x(0) = \xi, \tag{4.3}$$

and the condition

$$\int_0^T \varphi(s)x(s)ds = \psi, \tag{4.4}$$

where (x, u) takes values in $X \times \mathcal{U}$, $z \in X$; $D_0^\alpha, \alpha \in (0, 1)$, is the fractional derivative in the Caputo sense. In this model, A is a closed linear operator on X, $\varphi \in C^1([0, T]; \mathbb{R})$ is a nonnegative nontrivial function, $B : \mathcal{U} \to \mathbb{R}$, $h : X \to X$, $F : X \to \mathcal{U}^*$ and $G : \mathcal{U} \to \mathcal{U}^*$ are given maps. The notation $\langle \cdot, \cdot\rangle$ stands for the canonical pairing between $\mathcal{U}$ and its dual $\mathcal{U}^*$.

Our purpose of this chapter is to find appropriate conditions on $A, B, F, G, h, \xi, \psi, \varphi$, which ensure solvability, uniqueness and Lipschitz stability for the problem (**FrIP**) (4.1)-(4.4).

4.2. Solvability

In order to deal with the problem (**FrIP**) (4.1)-(4.4), we make use of the following assumptions:

(**A**) The operator A is sectorial and generates a compact semigroup such that

$$\|S(t)v\| \leq e^{-\beta t}\|v\|, \ \forall t \geq 0, v \in X,$$

where β is a positive number.

(**B**) The function $B : \mathcal{U} \to \mathbb{R}$ is Lipschitzian, that is, there exists a positive constant $L_B > 0$ such that

$$\|B(u_1) - B(u_2)\| \leq L_B\|u_1 - u_2\|_{\mathcal{U}}, \forall u_1, u_2 \in \mathcal{U}.$$

Moreover, there exist $m_B, M_B > 0$ such that $m_B \leq B(v) \leq M_B$ for all $v \in \mathcal{K}$.

(**F**) The operator $F : X \to \mathcal{U}^*$ is Lipschitz continuous with constant L_F, that is

$$\|F(y_1) - F(y_2)\|_{\mathcal{U}^*} \leq L_F \|y_1 - y_2\|, \ \forall y_1, y_2 \in X.$$

(**G**) The operator $G : \mathcal{U} \to \mathcal{U}^*$ is defined by

$$\langle G(u), v \rangle = b(u, v), \forall u, v \in \mathcal{U},$$

where $b : \mathcal{U} \times \mathcal{U} \to \mathbb{R}$ is a bilinear continuous function on $\mathcal{U} \times \mathcal{U}$ such that

$$b(u, u) \geq \eta_G \|u\|_{\mathcal{U}}^2, \forall u \in \mathcal{U},$$

here η_G is a positive number.

(**H**) The function $h : X \to X$ is a Lipschitz function with constant L_h, that is

$$\|h(y_1) - h(y_2)\| \leq L_h \|y_1 - y_2\|, \ \forall y_1, y_2 \in X.$$

Put

$$\mathcal{U}_{ad} = \{u \in C([0, T]; \mathcal{U}) : \ u(t) \in \mathcal{K}, \forall t \in [0, T]\}.$$

Definition 4.1. For given $\xi, z \in X$, a pair $(x, u) \in C([0, T]; X) \times \mathcal{U}_{ad}$ is called a classical (integral) solution of the problem (4.1)-(4.3) if (x, u) satisfies (4.2) and x is a classical (integral) solution of the problem

$$\begin{aligned} D_0^\alpha x(t) &= Ax(t) + f(t), \ t \in (0, T], \\ x(0) &= \xi, \end{aligned}$$

with $f(t) = B(u(t))z + h(x(t))$.

Definition 4.2. For given $\xi, \psi \in X$, a triple $(x, u, z) \in C([0, T]; X) \times \mathcal{U}_{ad} \times X$ is said to be a solution of the problem (**FrIP**) if (x, u) is a classical solution of the problem (4.1)-(4.3) and x fulfills (4.4).

We are in a position to state the main result of this section.

Theorem 4.1. *Let* (**A**)*,* (**B**)*,* (**F**)*,* (**G**)*,* (**H**)*, and* (4.5) *hold. Assume that* $\xi \in D(A)$*. Then the problem* (**FrIP**) *has a solution, provided that*

$$\frac{M_B(L_\varphi + L_h \|\varphi\|_{L^1})}{m_B \|\varphi\|_{L^1}} + L_h < \beta, \tag{4.5}$$

where $L_\varphi = \dfrac{T^{1-\alpha}}{\Gamma(2-\alpha)}(|\varphi(T)| + T \sup_{t \in [0,T]} |\varphi'(t)|)$.

4.3. Uniqueness and Stability

In this section, we consider the correspondence $(\xi, \psi) \mapsto (x, u, z)$, where (x, u, z) is the solution of (**FrIP**) with respect to (ξ, ψ).

Lemma 4.1. *Assume that the hypotheses* (**A**)*,* (**B**)*,* (**F**)*,* (**G**)*,* (**H**)*, and* (4.5) *hold. Let* (x, u, z) *be a solution of* (**FrIP**) *with respect to the pair* (ξ, ψ)*. Then there exists* $\rho_0 = \rho_0(\xi, \psi) > 0$ *such that* $\|x\|_\infty \le \rho_0$*. In addition, if* $(\hat{x}, \hat{u}, \hat{z})$ *is another solution of* (**FrIP**) *with respect to* $(\hat{\xi}, \hat{\psi})$*, then one can find positive numbers* δ *and* $\rho = \rho(\xi, \psi, \hat{\xi}, \hat{\psi})$ *such that*

$$\|x - \hat{x}\|_\infty \le \rho \left(\|\xi - \hat{\xi}\| + \|\psi - \hat{\psi}\|_{D(A)} \right),$$

provided that $L_F < \delta$.

The uniqueness and stability of solution to (**FrIP**) are given in the following theorem.

Theorem 4.2. *Assume that the hypotheses* (**A**)*,* (**B**)*,* (**F**)*,* (**G**)*,* (**H**)*, and* (4.5) *hold. Then the solution of the problem* (**FrIP**) *is unique for each pair* (ξ, ψ)*. Moreover, there exists* $\delta > 0$ *such that the solution map* $(\xi, \psi) \to (x, u, z)$ *is locally Lipschitz continuous as a map from* $X \times D(A)$ *to* $C([0, T]; X) \times C([0, T]; \mathcal{U}) \times X$*, provided that* $L_F < \delta$*.*

4.4. Application

Let Ω be a bounded domain in $\mathbb{R}^d, d \ge 1$, with smooth boundary $\partial\Omega$. Assume that $\varrho \in H^2(\Omega)$ is a nonnegative function. Let

$$\mathcal{K} := \{v \in L^2(\Omega) : v(x) \ge \varrho(x) \text{ for a.e. } x \in \Omega\}.$$

We consider the following problem: find $z \in L^2(\Omega)$ and $Y, u \in C([0, T]; L^2(\Omega))$ satisfying

$$\partial_t^\alpha Y(t, x) = \Delta_x Y(t, x) + z(x) \int_\Omega Q(y, u(t, y))dy + \tilde{h}(x, Y(t, x)), x \in \Omega, \tag{4.6}$$

$$- \Delta_x u(t, x) + W(u(t, x) - \varrho(x)) \ni f(x, Y(t, x)), x \in \Omega, \tag{4.7}$$

$$Y(t, x) = u(t, x) = 0, x \in \partial\Omega, \tag{4.8}$$

$$Y(0, x) = \xi(x), \ x \in \Omega, \tag{4.9}$$

and the measurement condition

$$\frac{1}{T} \int_0^T Y(t, x)\, dt = \psi(x), x \in \Omega, \tag{4.10}$$

where ∂_t^α stands for the Caputo derivative of order $\alpha \in (0, 1)$ in the time variable, $W : \mathbb{R} \to 2^{\mathbb{R}}$ is a maximal monotone graph,

$$W(r) = \begin{cases} 0 & \text{if } r > 0, \\ \mathbb{R}^- & \text{if } r = 0, \\ \emptyset & \text{if } r < 0. \end{cases}$$

The function φ is nonnegative and differentiable on $[0, T]$, $\xi, \psi \in H^2(\Omega) \cap H_0^1(\Omega)$, the functions $Q, \tilde{h}$ and f will be described later.

Let $X = \mathcal{U} = L^2(\Omega)$. The norm in X and $\mathcal{U}$ is given by

$$\|u\|^2 = \int_\Omega |u(x)|^2\, dx.$$

Define the function

$$B : \mathcal{U} \to \mathbb{R},$$
$$B(v) = \int_\Omega Q(y, v(y))dy. \tag{4.11}$$

Then (4.6) can be represented as

$$\partial_t^\alpha Y(t) = AY(t) + B(u(t))z + h(Y(t)), t \in [0,1],$$

where $A = \Delta, D(A) = H_0^1(\Omega) \cap H^2(\Omega), Y(t) \in X, u(t) \in \mathcal{U}$ such that $Y(t)(x) = Y(t,x), u(t)(x) = u(t,x)$, and $h : X \to X$ such that

$$h(v)(x) = \tilde{h}(x, v(x)),\ x \in \Omega. \tag{4.12}$$

It is known that A is the infinitesimal generator of a compact, analytic semigroup $\{S(t)\}_{t\geq 0}$ on X. Furthermore, we have

$$\|S(t)\|_{op} \leq e^{-\lambda_1 t},\ \forall t \geq 0,$$

where λ_1 is given by

$$\lambda_1 = \sup_{\|u\|=1} \|\nabla u\| > 0.$$

We now consider (4.7). Put $G = -\Delta$, where $-\Delta$ denotes the distributional Laplace operator, namely,

$$\langle -\Delta u, v\rangle := \int_\Omega \nabla u(x) \nabla v(x) dx, \text{ for all } u, v \in H_0^1(\Omega).$$

Thus $\langle Gu, u\rangle = \|u\|^2_{H_0^1(\Omega)} \geq \lambda_1 \|u\|^2$. So the assumption (**G**) is fulfilled with $\eta_G = \lambda_1$.

Let $F : X \to X$ be the map defined by

$$F(v)(x) = f(x, v(x)),\ x \in \Omega. \tag{4.13}$$

Noting that, inclusion (4.7) can be written as

$$-\Delta u(t) + \partial I_{\mathcal{K}}(u(t)) \ni F(Z(t)),$$

where

$$\begin{aligned}\partial I_{\mathcal{K}}(u) &= \{v \in L^2(\Omega) : \int_\Omega v(x)(u(x) - z(x))\, dx \geq 0, \forall z \in \mathcal{K}\}\\ &= \{v \in L^2(\Omega) : v(x) \in W(u(x) - \varrho(x)),\ \text{for a.e. } x \in \Omega\}.\end{aligned}$$

Regarding the nonlinear functions appearing in (4.6)-(4.7), we assume that Q, $\tilde{h}$ and f are Carathéodory functions defined on $\Omega \times \mathbb{R}$ such that:

(N1) The function $Q(x, \cdot)$ is nondecreasing and

$$Q(x, r) \leq p(x),\ \forall (x, r) \in \Omega \times \mathbb{R},$$

where $p \in L^1(\Omega)$ is a nonnegative function;

(N2) $|Q(x, r) - Q(x, s)| \leq q(x)|r - s|$, (N3) $|\tilde{h}(x,r) - \tilde{h}(x,s)| \le k(x)|r-s|$,

(N4) $|f(x,r) - f(x,s)| \le \ell(x)|r-s|$,

for all $x \in \Omega$ and $r, s \in \mathbb{R}$, here $q, k, \ell \in L^2(\Omega)$ are given functions. Then it follows from (N1)-(N2) that the function B given by (4.11) obeys the following estimates

$$
\begin{aligned}
B(v) \le M_B &:= \int_\Omega p(x)dx, \ \forall v \in X \\
B(v) \ge m_B &:= \int_\Omega Q(y, \varrho(y))dy, \ \forall v \in \mathcal{K}, \\
|B(v_1) - B(v_2)| &\le L_B \|v_1 - v_2\|, \ \forall v_1, v_2 \in X,
\end{aligned}
$$

where $L_B = \|q\|$. Thus (**B**) is satisfied, provided that $m_B > 0$.

It is easily seen that the function h defined by (4.12) is Lipschitzian, that is

$$
\|h(v_1) - h(v_2)\| \le L_h \|v_1 - v_2\|, \ \forall v_1, v_2 \in X,
$$

where $L_h = \|k\|$, thanks to (N3). Similarly, the function F defined by (4.13) is a Lipschitz function with $L_F = \|\ell\|$ due to (N4). So (**H**) and (**F**) are fulfilled. According to Theorem 4.1 and 4.2, we can state the following result.

Theorem 4.3. *Let the assumptions* (N1)-(N4) *hold. Then the problem* (4.6)-(4.10) *has a unique solution and the map* $(\xi, \psi) \mapsto (Y, u, z)$ *is locally Lipschitzian as a map from* $X \times D(A)$ *to* $C([0,T]; X) \times C([0,T]; X) \times X$, *provided that*

$$
M_B(T^{-\alpha}\Gamma(2-\alpha)^{-1} + \|k\|) + m_B\|k\| < \lambda_1 m_B,
$$

and $\|\ell\|$ *is small enough.*

Remark 4.1. It should be noted that, the system (4.6)-(4.9) is a generalized parabolic-elliptic problem. If we choose $\mathcal{K} = L^2(\Omega)$, then it is easily seen that $\partial I_{\mathcal{K}}(u) = \{0\}$. So it follows from (4.7) that

$$
u(t,x) = \mathbb{V}(Y)(t,x) = (-\Delta_x)^{-1} f(x, Y(t,x)).
$$

Using the last relation for (4.6), we get the equation of the form

$$
\partial_t^\alpha Y = \Delta Y + \mathcal{Q}(Y)z + h(Y),
$$

and our question turns out to an identification problem for time-fractional diffusion equations.

CONCLUSION

1. Results of the thesis

This thesis has studied some qualitative properties for some classes of NDEs, including: the finite-time attractivity, the stability of solutions; the solvability, the uniqueness and Lipschitz stability for the problem of identifying parameter. Our results:

(a) For semilinear subdiffusion equations and Basset type evolution equations considered in a bounded time-interval, we obtained:

- The results of the solvability with the nonlinear part having linear growth; The finite-time attractivity of zero solution and of nonzero solutions;
- Applications of our results to a semilinear subdiffusion equation and to a semilinear integrodifferential equation.

(b) For a class of semilinear Rayleigh-Stokes type evolution equations, we obtained:

- The results of the solvability with the nonlinear part having linear growth;
- The asymptotic stability of the zero solution; The existence of decay solutions.

(c) For the problem of identifying parameter in a class of fractional differential variational inequalities, we obtained:

- The solvability, the uniqueness and the Lipschitz stability;
- An application to a generalized parabolic-elliptic problem.

2. Recommendation

Beside the results obtained in this thesis, some open problems are:

- Study the stability and weak stability for nonlocal differential equations involving delays or impulsive;
- Study the solvability, the stability for inverse problems governed by nonlocal differential equations;
- Study the regularity and the convergence to equilibrium point of solutions for nonlocal differential equations.

AUTHOR'S WORKS RELATED TO THE THESIS THAT HAVE BEEN PUBLISHED

1. ..., ..., (2018), Finite-time attractivity for semilinear fractional differential equations, *Results Math.*, 73:7, 19 pp.

2. ..., (2020), Short-time behavior for a class of semilinear nonlocal evolution equations in Hilbert spaces, *Appl. Anal. Optim.*, accepted.

3. ..., ..., (2020), An identification problem involving fractional differential variational inequalities, *J. Inverse Ill-Posed Probl.*, doi: 10.1515/jiip-2017-0103, accepted.

4. T.D. Ke, T.V. Tuan, (2020), Stability analysis for a class of semilinear nonlocal evolution equations, submitted.

Results of the thesis have been reported at:

- Seminar of Department of Analysis, Faculty of Mathematics, Hanoi Pedagogical University 2;
- Seminar of Department of Analysis, Faculty of Mathematics, Hanoi National University of Eduacation;
- Conference for PHD students, Faculty of Mathematics, Hanoi Pedagogical University 2, 2017, 2018, 2019.