MINISTRY OF EDUCATION AND TRAINING
HA NOI PEDAGOGICAL UNIVERSITY 2

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STUDY OF THE CASIMIR EFFECT IN BOSE-EINSTEIN CONDENSATE

Major: Theoretical and Mathematical Physics
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SUMMARY OF DOCTORAL THESIS IN PHYSICS

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Introduction

1. Motivation

More than 50 years since H.B.G. Casimir published his famous work[16],in which, he gave a simple but profound explanation for the restarted van der Waals interaction (this interaction was described by him with D. Polder just a short time earlier) as an expression of zero energy of a quantized field. For a long time, the paper was not clearly known. But starting from 1970s and later that physicists began to pay attention to the Casimir effect and studied it in different physical systems. New high precision experiments on the demonstration of the Casimir force have been performed and more are underway. In theoretical developments, significant progress had been made to the investigation of the structures of the divergencies in general, non-flat background, and in the calculation of the effect for more complicated geometries and boundary conditions including those due to the real structures of the boundaries. The Casimir effect is an interdisciplinary problem, its application scope is very wide, from cosmology to the physics of solidified environments, especially nano physics and nanomaterial fabrication technology.

Although predicted since 1925, studies of Bose-Einstein condensate (BEC) only really exploded since 1995 in both the theory and experiment, especially after experimentally created the two-component Bose-Einstein condensates (BECs). However, after the experimental successes of measuring the critical Casimir force in quantum liquids and the Casimir-Polder force in BEC medium, the studies of the finite size effect in BEC system really explode. Because the BEC can be considered as a quantum liquid, there exists a surface energy that corresponds to the appearance of a Casimir-like force, which is considered as the average field component of the Casimir force. For this reason, the effects of space shrinkage in the BEC are often studied in two aspects: The first issue is that the surface tension caused when the BEC is confined between two parallel plates, and this is called the Casimir-like effect (which corresponds to a Casimir-like force); the second one is that the effect caused by quantum fluctuations on top of ground state, which corresponds to phonic excitations

[16]Casimir H. B. G. (1948), "On the attraction between two perfectly conducting plates", *Proc. K. Ned. Akad. Wet.* **51**, 793.

and also known as the quantum fluctuation component of the Casimir force.

Firstly, we mention the Casimir-like effect. For the ideal Bose gas, based on the Bose-Einstein statistics, S. Biswas(12) calculated the Casimir-like force for different regions of temperature, including those much greater than the critical temperature. For the dilute Bose gas, the Casimir-like force calculation was performed by the S. Biswas and D. Roberts research groups. N.V. Thu et al(74) investigated Casimir-like force of the two-component BECs in double parabolic approximation (DPA) in the demixing critical state.

For the Casimir force caused by quantum fluctuations, the available studies are relatively rich. However, some of the limitations of these studies are:

- The calculations can only be performed in one-loop approximation;

- Considering only with a single BEC. For two-component BECs, due to the appearance of interactions between particles in two different components, thus many important results will be found. However, as our knowledge, the field has been still absent so far;

- Considering only in grand canonical ensemble (GCE).

It is clear that there are still many problems regarding the effect of spatial compactification on the properties of BEC that need to be further studied. In order to contribute orientation to the experimental studies on the Casimir effect in BEC, we choose **"Study of the Casimir effect in Bose-Einstein condensate"** as the research topic in this thesis.

2. Purpose, objectives and scopes

The influences of the finite-size effect on properties of the BEC confined between two parallel plates are considered. In this thesis, we focus on studying the BEC at zero temperature and without an external field in both the canonical ensemble (CE) and grand canonical ensemble. The details are in order:

a) A single BEC:

- Investigating the wave function describing the ground state based on the Gross-Pitaevskii equation (GP). After that, we find the surface tension energy and the Casimir-like force.

- Studying the influences of finite-size effect on the condensate density, Casimir energy and Casimir force in the one-loop and two-loop approximations.

- Studying the total Casimir force, which is the net force of the quantum

(12)Biswas S. (2007), "BoseEinstein condensation and the Casimir effect for an ideal Bose gas confined between two slabs", *J. Phys. A* **40**, 9969.

(74)Thu N. V., Phat T. H., Song P. T. (2017), "Finite-size effects of surface tension in two segregated BECs confined by two hard walls", *Journal of Low Temperature Physics.* **186**, 127.

Casimir force and the Casimir-like force.

b) Two-component BECs:

In case of the two-component BECs, due to the complex of the mathematical calculations, we only investigate in the GCE and mainly focus on:

- Using double parabola approximation to study the surface tension energy and Casimir-like force.

- Investigating the Casimir effect in the one-loop and two-loop approximations.

3. Research methods

In order to do the above studies, we choose the following research methods:

- When studying surface tension force, we use the mean field theory for the BEC at zero temperature, which is described by the GP(s) equation(s). In order to analytically solve this/these equation(s), we use the DPA.

- The Cornwall-Jackiw-Tombolis (CJT) effective action approach is employed to consider the Casimir effect.

4. Structure of the thesis

Besides the parts of introduction, conclusions, and references, the thesis includes:

Chapter 1. Theoretical research on Casimir force

Chapter 2. Casimir force of a single Bose-Einstein condensate

Chapter 3. Casimir force of two-component Bose-Einstein condensates

Chapter 1

Theoretical research on Casimir force

In this chapter, we try to systematize the content related to the Casimir effect in physics, especially in the Bose-Einstein condensate. At the same time, we present two basic methods widely used to study the Casimir effect in a single BEC and two-component BECs.

1.1. Overview of Casimir force

1.1.1. Zero-point oscillations and their manifestation

Zero-point energy is the lowest possible energy that a quantum mechanical system may have. The kinetic energy of atoms and molecules is proportional to the absolute of temperature, thus the temperature is reduced to absolute zero, all motion ceases and molecules come to rest. However, the quantum system is governed by Heisenberg's uncertainty principle, so atoms and molecules vibrate even at zero temperatures. The energy of system is now called zero-point energy.

For seek of simplicity, considering a one-dimensional harmonic oscillator with angular frequency ω. In the n stop state, its energy is

$$E_n = \hbar\omega\left(n + \frac{1}{2}\right), \tag{1.1}$$

with $\hbar$ is the reduced Planck's constant and $n = 0, 1, \ldots$ Thus, the ground state energy of this harmonic is

$$E_0 = \frac{\hbar\omega}{2} \neq 0. \tag{1.2}$$

This is called zero-point energy. The first experimental evidence for the existence of zero-point energy was observed by Mulliken in 1924. Thus, even at zero temperature, the atoms and molecules are always vibrating, which creates quantum fluctuations.

1.1.2. Casimir effect

In 1948, H.B.G. Casimir discovered interactions between two flat, electrically neutral plates placed parallel to each other in the electromagnetic field. This force is the Casimir force, it has form

$$f = -\frac{\pi^2 \hbar c}{240 \ell^4} S, \tag{1.20}$$

in which $\hbar$ and c are Planck constant and speed of light in vacuum, respectively, ℓ is the distance between the plates, S is their area and satisfies the condition $\ell^2 \ll S$.

To derive this result, there are different methods to cancel divergence in zero energy, but there are two common methods: the momentum cut-off and the Riemann zeta function methods.

Studies have shown that the Casimir force depends on many factors: the nature of the system, the geometrical structure of the system, boundary conditions and temperature. In experiments, accurately measuring the Casimir force is very difficult because: firstly, this force only appears in a very small region of space; secondly, it creates a structure like Casimir's original calculation is very difficult happening. There have been many experimental efforts to study the Casimir force but the results have not been as expected. It was not until 1996, that is, 48 years after being discovered, that Lamoreaux could measure the Casimir force with an error of 5 % compared with theoretical calculations.

1.2. Situation of studying Casimir force in Bose-Einstein condensate

1.2.1. Bose-Einstein condensate

Bosons are systems of identical particles with integer spins and obey the Bose-Einstein statistics. When the temperature T of the system is smaller (but very close to) than the critical temperature T_C^0, the number of condensate particles is

$$N_0 = N \left[1 - \left(\frac{T}{T_C^0} \right)^{\alpha} \right]. \tag{1.18}$$

1.2.2. Overview of studying Casimir effect in Bose-Einstein condensate

The study of the finite size effect in BEC was performed by Harber et al in 2005 to determine the Casimir-Polder force experimentally.

In terms of theory, the first work can be mentioned is A. Edery's research with three-dimensional BEC. By using the quantum field theory in the one-loop approximation, Schiefele and Henkel invoked Andersens results within framework of perturbative theory to consider the finite-size effect on BEC at zero and finite temperature. Their main result is that the Casimir force is attractive and decays as the distance L between two plates increases, which obeys the law L^{-4}. However, their results could not give a general law because they only considered in the critical regions, where distance is large/small enough. Besides, the authors consider only in the GCE.

Another research group is S. Biswas et al used Hamiltonian formalism to investigate the interactive forces in the BEC. The result of this work is to find the analytical function of the surface tension force caused by the excess energy per unit area in the mean field theory and the Casimir force. By the way, the authors evaluated the relation between the Casimir and surface tension forces. Momentum integral is ultraviolet divergence, so a momentum cut-off Λ is introduced for upper limit of these momentum integrals. In addition, the obtained results must be take a limit $\Lambda \to \infty$. It is unreasonable to expand momentum and limit to the fourth order above. To overcome this problem, N.V. Thu used Euler-Maclaurin formula to avoid ultraviolet divergencies. At the same time, expanding research in terms of boundary conditions and considering the system in GCE and CE. However, the common of these works is that system is considered in one-loop approximation so that the order parameter is independent on distance between two palates and the BEC is not been thoroughly considered.

In case of two-component BECs, Casimir-like effect (correspond to Casimir-like force) was studied in the demixing critical state by the authors N.V. Thu and et al. Consequently, the Casimir-like force do not vanish at demixing limit. This is a very special result of finite size effect on the static properties of the two-component BECs. To our understanding, the study of the Casimir effect in two-component BECs have been still absent so far.

In order to take part into further clarification on the influences of finite size effect on the static properties of two-component BECs, the main aim of this thesis is concentration on the two most important quantities which are the Casimir-like and Casimir forces.

1.2.3. The Gross-Pitaevskii theory

a. The Gross-Pitaevskii equation

The GP equation is essentially a nonlinear form of the Schordinger equation when considered in the mean field theory. Considering a single BEC without external field, the Gross-Pitaevskii equation has the form

$$\left[-\frac{\hbar^2}{2m}\nabla^2 - \mu + g|\Psi(\vec{r})|^2\right]\Psi(\vec{r}) = 0. \tag{1.62}$$

- The GP interaction potential is written

$$V_{GP} = -\mu|\Psi|^2 + \frac{g}{2}|\Psi|^4. \tag{1.63}$$

b. The Gross-Pitaevskii equations

For a two-component BECs, the interaction between pairs of particles consists of two different components: the first is the interaction between pairs of particles in the same component, which is characterized by the interaction constant g_{jj}; the second is the interaction between pairs of particles in two different components with the interaction constant g_{12}.

- Applying the principle of minimum action, we find the time-independent GP equations

$$\begin{aligned} -\frac{\hbar^2}{2m_1}\nabla^2\Psi_1 - \mu_1\Psi_1 + g_{11}|\Psi_1|^2\Psi_1 + g_{12}|\Psi_2|^2\Psi_1 = 0, \\ -\frac{\hbar^2}{2m_2}\nabla^2\Psi_2 - \mu_2\Psi_2 + g_{22}|\Psi_2|^2\Psi_2 + g_{12}|\Psi_1|^2\Psi_2 = 0. \end{aligned} \tag{1.69}$$

- The GP interaction potential has form

$$V_{GP} = \sum_{j=1,2}\left[-\mu_j|\Psi_j|^2 + g_{jj}|\Psi_j|^4\right] + g_{12}|\Psi_1|^2|\Psi_2|^2. \tag{1.70}$$

1.2.4. The double parabola approximation

To understand the properties of a single BEC and two-component BECs, the first requirement is to solve the GP equation (equations). Because the GP equation (equations) is nonlinear differential equations, to find the analytical solution, we use the DPA proposed by Joseph et al in 2015. By expanding the order parameter aroung its bulk value and keeping up to second order of wave function in the ground state, one has DPA potential in a dimensionless form

$$\mathcal{V}_{DPA} \approx 2(\phi - 1)^2 - \frac{1}{2}. \tag{1.78}$$

As a result, the Euler-Lagrange equation in dimensionless form is achieved

$$-\frac{\partial^2\phi}{\partial\rho^2} + \alpha^2(\phi - 1) = 0, \tag{1.79}$$

with $\alpha = \sqrt{2}$.

1.2.5. The Cornwall-Jackiw-Tombolis (CJT) effective action

The Cornwall-Jackiw-Tombolis (CJT) effective action approach is employed to consider the Casimir effect. Therefore, the state of system is determined through:

- The gap equation

$$\frac{\partial V_{eff}(\bar{\phi}, G)}{\partial \bar{\phi}} = 0. \tag{1.89a}$$

- Schwinger-Dyson (SD) equation

$$\frac{\partial V_{eff}(\bar{\phi}, G)}{\partial G} = 0. \tag{1.89b}$$

When equations (1.89) for the solution $\bar{\phi}(x) \neq 0$, which means that there is symmetry breaking. Thus, spontaneous symmetry breaking was automatically generated in the CJT effective-effect formalism.

Chapter 2

The Casimir effect in a single Bose-Einstein condensate

In Chapter 2, the Casimir effect in a single BEC confined between two parallel palates is studied. Let the $0z$ axis be perpendicular to the two plates and the positions of coordinate origin is between two plates. We consider the system in both GCE and CE. Besides, studying the Casimir force in the two-loop approximation, some problems have not been completely solved in the work of the author N.V. Thu[75] *will be investigated in this chapter. The contents of this chapter are reflected in the papers 3, 4 and 5 in the published works related to the thesis.*

2.1. Study of the Casimir-like force

We use Dirichlet boundary conditions, which are still widely used, and can be generated in experiments using optical or magnetic potentials to study the Casimir-like force in a single BEC.

2.1.1. The ground state

To begin, we find the wave function of ground state, ie find the solution of the equation GP in the DPA. The Dirichlet boundary conditions in dimensionless form

$$\phi\left(-\frac{L}{2}\right) = \phi\left(\frac{L}{2}\right) = 0, \tag{2.2}$$

with $L = \ell/\xi$. Combined with the Euler-Lagrange equation in dimensionless form, we obtain the order parameter describing the wave function of the ground state.

$$\phi(\rho) = 1 - \cosh(\alpha\rho)\sec\left(\frac{L}{\alpha}\right). \tag{2.3}$$

[75] Thu N. V. (2018), "The forces on a single interacting BoseEinstein condensate." *Physics Letters A* **382**, 1078.

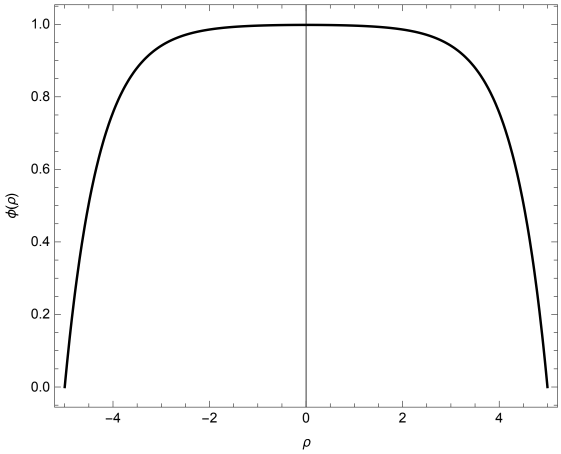

Figure 2.1: The order parameter depends on coordinates.

As a result, the wave function describing the ground state of the system is vanished at the positions of the plates and quickly reaches the bulk value in the limited space between the two parallel palates.

2.1.2. The Casimir-like force in the GCE

We now use the wave function to investigate the surface tension energy, from that we find Cassimir-like force in the GCE and the distance evolution of the Casimir-like force are shown in Fig 2.3

$$F_\gamma = -4P_0 \text{sech}\left(\frac{L}{\alpha}\right)^2. \tag{2.11}$$

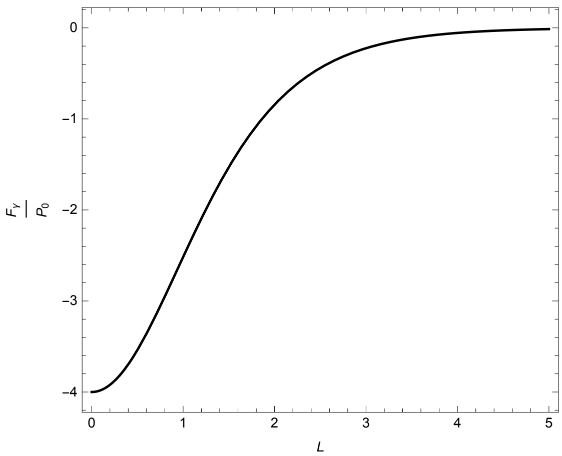

Figure 2.3: The Casimir-like force versus L in GCE.

It is obvious that this force is attractive and its strength tends to zero at large-L region, the finite-size effect becomes faint. When the distance approaches 0, the Casimir-like becomes a constant $-4P_0$.

2.1.3. The Casimir-like force in CE

For a single BEC in the CE, this means that, BEC is not connected to a particle reservoir, thus, particle number N is fixed. Based on the wave of ground state, we find analytical expression of the Casimir-like force in the CE

$$\frac{F_\sigma}{F_0} = \frac{F_1}{4g^2\ell m^2 N^2 \left[-3\alpha\xi \sinh\left(\frac{\alpha\ell}{\xi}\right) + 2\ell \cosh\left(\frac{\alpha\ell}{\xi}\right) + 4\ell\right]^2}, \tag{2.21}$$

with

$$F_1 = S\hbar^2 \left\{ 2\alpha\xi \sinh\left(\frac{\alpha\ell}{\xi}\right) \left[3S\hbar^2 \cosh\left(\frac{\alpha\ell}{\xi}\right) + 9S\hbar^2 - 4glmN\right] - 4S\ell\hbar^2 \cosh\left(\frac{\alpha\ell}{\xi}\right) \right\},$$

and $F_0 = \frac{2m^2 g^3 N^4}{\hbar^4 S^4}$.

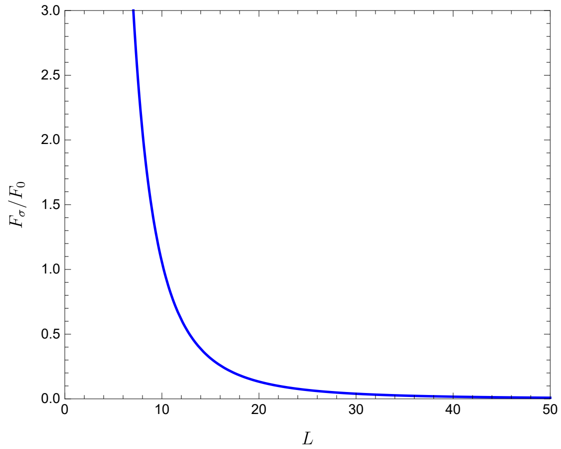

Figure 2.5: The Casimir-like force versus L in CE.

Fig 2.5 shows the Casimir-like force in the CE versus L. It is obvious that, the same result in the GCE, this force tends to zero when the distance is large enough to disregard the finite size effect. There are essential different points in comparing to the one in the GCE, this force is repulsive and strongly depends on L. As L tends to zero this force becomes infinity, the reason is the incompressibility of condensate.

2.2. The Casimir force

In this section, we study the influence of finite size effect on the Casimir energy and Casimir foce in the weakly interacting BEC in Hartree-Fock approximation (HF), including quantum fluctuations. Based on the CJT effective action, we study the influence of finite-size effect on a single BEC in the two-loop approximation. In the meantime, investigating the total Casimir force acts on slabs, which consists of two components, namely, Casimir-like force caused by excess surface energy and Casimir force corresponding to the quantum fluctuation, are independent in both GCE and CE.

2.2.1. Studying in the one-loop approximation

When considering in the one-loop approximation, we use a momentum cut-off by introducing an upper limit Λ to cancel the ultraviolet divergences UV. As a consequence, Casimir energy has form

$$\Omega = -\frac{gn_0}{\xi^2}\frac{\pi^2\phi}{1440L^3}. \tag{2.45}$$

The Casimir force is defined as the negative first derivative of the Casimir energy according to the distance between two slabs

$$F_C = -\frac{\partial\Omega}{\partial\ell}. \tag{2.46}$$

a. In the GCE

In the GCE, because the bulk density of condensate is a constant, thus healing length ξ is also constant. The Casimir force in the GCE has the form

$$F_C = -\frac{gn_0}{\xi^2}\frac{\pi^2\phi}{480L^4}. \tag{2.48}$$

The total Casimir force acts on slabs is

$$F_{\text{total}} = F_\gamma + F_C. \tag{2.51}$$

It is obvious from (2.48), Fig 2.6, and Fig 2.7, we have following comments: Firstly, Casimir-force is always attractive, thus it enhances the strength of total force acting on the palates; the another point is that there is a divergence at $L = 0$, this is typical characteristic of distort of the vacuum energy; last but not least, the strength decays sharply as distance increases, this means that Casimir force is noticeable at small distance.

b. In the CE

When considering in the CE, bulk density of condensate n_0 depends on distance between the two parallel plates, thus healing length also depends on

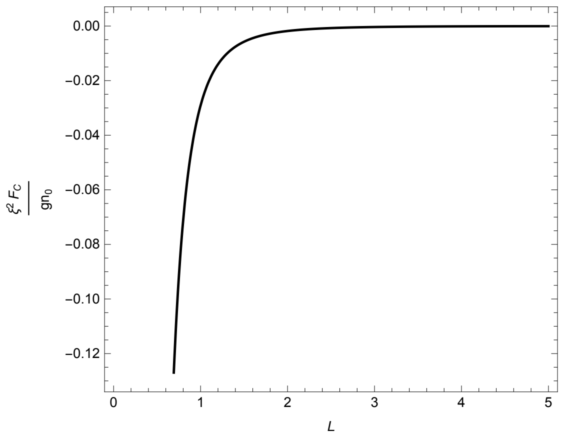

Figure 2.6: The Casimir force versus L in GCE.

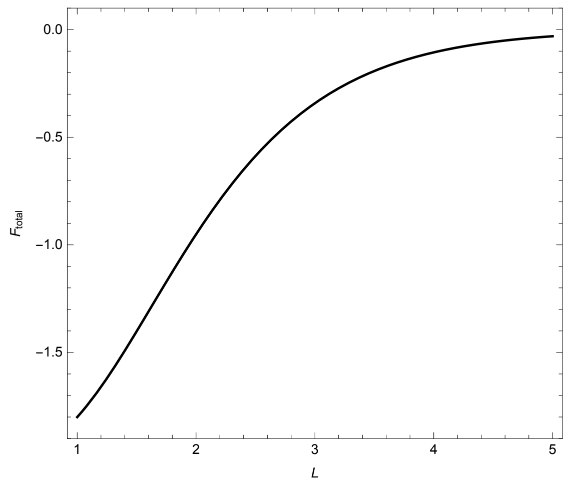

Figure 2.7: The total Casimir force versus L in GCE.

ℓ. Therefor, the energy Casimir (2.45) is rewritten in dimension form

$$\Omega = -\frac{\pi^2 \phi \hbar^2}{1440 \alpha m \xi I_0 \ell^2}. \tag{2.53}$$

As a result, we obtained the Casimir force in the CE

$$\frac{F_C}{F_0} = \frac{M}{1440 \alpha m^3 g^3 N^4 \ell^3 \left[2\ell \cosh\left(\frac{\alpha \ell}{\xi}\right) + 4\ell - 3\alpha\xi \sinh\left(\frac{\alpha\ell}{\xi}\right)\right]^2}, \tag{2.54}$$

in which

$$\begin{aligned} M =& \pi^2 S^3 \phi \hbar^4 \cosh\left(\frac{\ell}{\alpha\xi}\right)\left[\sinh\left(\frac{\ell}{\alpha\xi}\right)\left(9S\hbar^2 + 4gmN\ell\right)\right. \\ &\left. + 9S\hbar^2 \sinh\left(\frac{3\ell}{\alpha\xi}\right) - 29\alpha mgN\xi \cosh\left(\frac{\ell}{\alpha\xi}\right) - 7\alpha mgN \cosh\left(\frac{3\ell}{\alpha\xi}\right)\right]. \end{aligned}$$

By the same way in the GCE, we obtained the total Casimir force in the CE

$$F_{\text{total}} = F_\sigma + F_C. \tag{2.55}$$

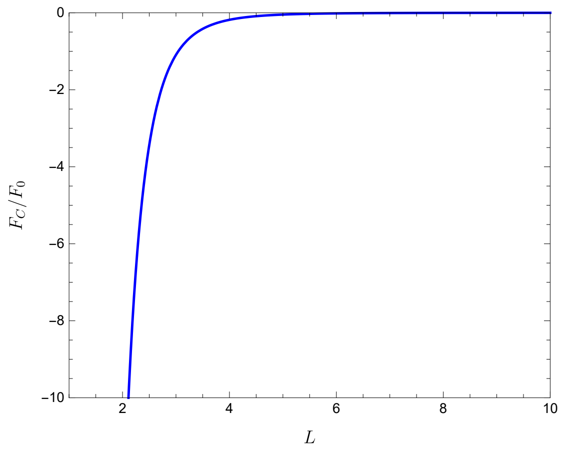

Figure 2.8: The Casimir force versus L in CE.

Let we now solve some remaining problems in the work of author N.V. Thu. First, like for the Casimir-like force, here numerical computations are performed for a single BEC of rubidium 87, instead of sodium 23. Fig 2.9 shows evolution of the total Casimir force versus distance. The result points out that the total force is repulsive in large distance region (red curve) and attractive in small distance region (blue curve). For rubidium 87, the total force changes it direction at point M with $L = 1.0327$. This point coincides exactly to the point at which total energy $E = \sigma + \Omega$ is maximum on Fig 2.10.

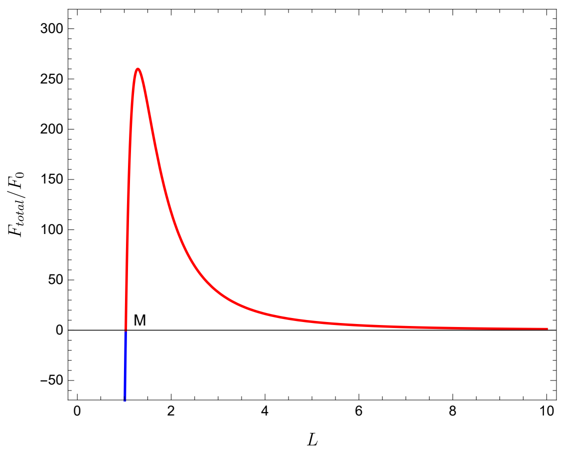

Figure 2.9: The total Casimir force versus L in CE.

Our result shows that Casimir force in the CE and GCE are only different to magnitude and decrease speed when the distance between two plates increases. A result has not been investigated in the work of author N.V. Thu is instead of proportional to the integer power law of the distance in the GCE, in CE the Casimir force is proportional to the half-integral power law of distance between the two parallel plates. Furthermore, considering in the CE, the Casimir-like and Casimir force (in dimensionless form) not only depends on the distance, but also each specific systems: particle number, atomic mass, and inter-atomic interaction.

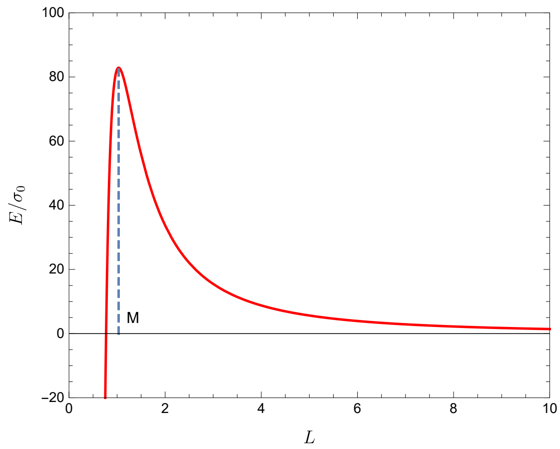

Figure 2.10: The total energy versus L in CE.

At the position where the total force vanishing, Casimir-like and Casimir forces have the same magnitude but opposite direction each other. Therefore, we find the value of distance between the two plates at which the veer of total Casimir force takes place

$$\ell_0 \approx \frac{1}{4}\left(\frac{121\pi^4\hbar^6 S^5}{450m^3g^3N^3}\right)^{1/7}. \tag{2.58}$$

Therefore, there always exists a value of the distance between two parallel plates where the Casimir force is completely vanished with different atomic systems.

2.2.2. Studying in the two-loop approximation

The previous studies of the Casimir force only are considered in the one-loop approximation. In order to investigate the influences of one-dimensional contraction on the static properties of the single BEC, we extend the research considering the high order approximation (two-loop approximation). To do so, we use the CJT effective action approach and add a term to the effective potential in the HF approximation, we obtained the SD and gap equations, which describe the state of a single BEC in dimensionless form

$$\begin{aligned} \mathcal{M} &= -1 + 3\phi^2 + \frac{3g}{2}\frac{m\mathcal{M}^{1/2}}{12\hbar^2\ell}, \\ -1 + \phi^2 &+ \frac{g}{2}\frac{m\mathcal{M}^{1/2}}{12\hbar^2\ell} = 0. \end{aligned} \tag{2.81}$$

Solving (2.81), one easily finds the analytical solution for order parameters and effective mass in dimensionless form

$$\mathcal{M} = 2, \tag{2.82}$$

$$\phi = \sqrt{1 + \frac{mg\mathcal{M}^{1/2}}{24\hbar^2 L\xi}}. \tag{2.83}$$

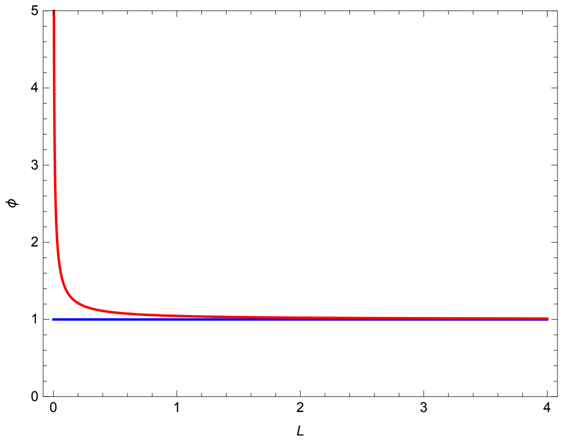

Figure 2.11: The order parameter as a function of the distance L in two-loop.

It is obviously that the effective mass is the same as that in the one-loop approximation while the order parameter is different. In the one-loop approximation, the order parameter is constant and equals to unity. Eq.(2.83) shows that in the improved HF approximation (IHF), the effective mass strongly depends on the distance between two plates, especially in small-L region and it turns out to be divergent when the distance approaches to zero. Because of independence of the effective mass M on distance between two plates, the finite-size effect have no extra contribution on Casimir force in comparing to the one in the one-loop approximation.

Chapter 3

The Casimir effect in two-component Bose-Einstein condensates

In this Chapter, we research the Casimir effect in two-component Bose-Einstein condensates, thus in addition the interspecies interaction, there are also the interspecies interaction with the interaction constant g_{12}. Due to this interaction, the two-component BECs can be in miscible or immiscible state. For immiscible case, the interspecies interaction causes lots of changes in the properties of system, especially in strong segregation. The contents of this chapter are reflected in the papers 1 and 2 in the published works related to the thesis.

3.1. Studying the Casimir-like force

We consider a two-component BECs in the equilibrium state confined between two parallel palates, these palates are perpendicular to $0z$ axis and separated at distance ℓ. We assume that the plates are located at $z = 0$ and $z = \ell$, the interface between two components is located at $z = \ell/2$. This means that our system is limited in a rectangle with the volume $V = \ell_x \ell_y \ell_z$ which satisfies condition $\ell_x, \ell_y \gg \ell$. Based on the Dirichlet boundary condition, we study the Casimir-like force of BECs in GCE by using DPA.

3.1.1. The gound state

- The dimensionless Dirichlet boundary condition

$$\phi_j(\rho = 0) = \phi_j(\rho = L) = 0, \text{vi}\, j = (1, 2). \tag{3.9}$$

- Using DPA method and the condition (3.9), we get the wave function of condensate

$$\phi_1 = e^{-\alpha\rho}\left(e^{\alpha\rho} - e^{\alpha L}\right)\left(A_1(e^{\alpha L} + e^{\alpha\rho}) + 1\right), \tag{3.10a}$$

$$\phi_2 = -2B_1 e^{\frac{L\beta}{\xi}} \sinh\left(\frac{\beta(L-\rho)}{\xi}\right), \tag{3.10b}$$

in the right hand side of interface,

$$\phi_1 = A_2\left(e^{\beta\rho} - e^{-\beta\rho}\right), \tag{3.11a}$$

$$\phi_2 = B_2\left(e^{\frac{\alpha\rho}{\xi}} - e^{\frac{-\alpha\rho}{\xi}}\right) + 1 - e^{\frac{-\alpha\rho}{\xi}}, \tag{3.11b}$$

in the left hand side of interface.

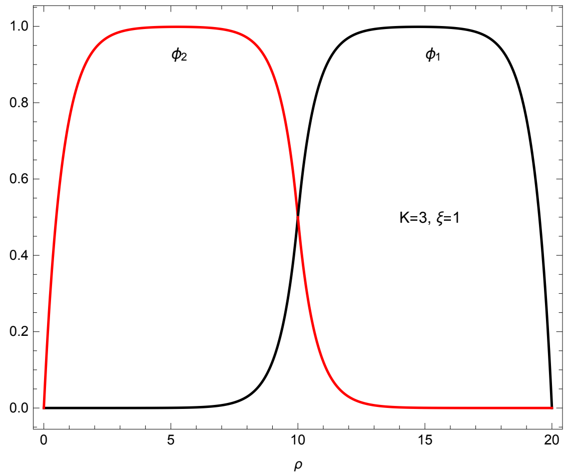

Figure 3.1: The wave function in ground state corresponds to $K = 3$ and $\xi = 1$.

- For the strong segregation $K \to \infty$

$$\phi_1 = \begin{cases} 1 - \cosh\left(\frac{3\alpha L}{4} - \alpha\rho\right)\operatorname{sech}\left(\frac{\alpha L}{4}\right), & \text{if } \rho > L/2; \\ 0, & \text{if } \rho < L/2. \end{cases} \tag{3.14}$$

$$\phi_2 = \begin{cases} 0, & \text{if } \rho > L/2; \\ 1 - \cosh\left(\frac{\alpha(L-4\rho)}{4\xi}\right)\operatorname{sech}\left(\frac{\alpha L}{4\xi}\right), & \text{if } \rho < L/2. \end{cases} \tag{3.15}$$

In this case, the ingredients will separate completely.

3.1.2. The Casimir-like force

To calculate the interface surface energy in our system. The system that we considered is different from the investigation system in the work of Joseph[(36)] and Schaeybroeck[(64)] as following:

- We examine a system of finite size in all three dimensions of space, especially along the $0z$ axis. This means that our system is similar in the work of N. V. Thu and et al [(74)];

[(36)]Indekeu J. O., Lin C. -Y., Thu N. V., Schaeybroeck V. B., Phat T. H. (2015), "Static interfacial properties of BoseEinstein-condensate mixtures", *Physical Review A* **91**, 033615.
[(64)]Schaeybroeck B. V. (2008), "Interface tension of Bose-Einstein condensates", *Physical Review A* **78**, 023624.
[(74)]Thu N. V., Phat T. H., Song P. T. (2017), "Finite-size effects of surface tension in two segregated BECs confined by two hard walls", *Journal of Low Temperature Physics.* **186**, 127.

-In the general case, the GP equations, as mentioned above, has no analytical solution. To be able to evaluate the Casimir-like force in different regions of the interaction constant K, the authors Schaeybroeck and N. V. Thu used "constant of motion" to take the kinetic term from the quadratic derivative to the first derivative of the wave function. In this framework, using the DPA, we find the analytical solution of the wave function, thus we can calculate the surface tension directly from its definition.;

- Due to the wave function in the DPA is determined for two different regions of interface, thus to calculate surface energy, we first calculate for each side of the interface and then perform the summing to get the result for whole system.

As a result, the interface tension has form

$$\begin{aligned}\gamma_{12} =& 2P\xi_1\alpha \left\{ \left[e^{\frac{L}{\alpha}} - 1\right] \left[1 + A_1 \left(e^{L\alpha} - e^{\frac{L}{\alpha}}\right)\right] \right. \\ & \left. + \xi \left[1 - e^{-\frac{L}{\alpha\xi}} - 4B_2 \sinh\left(\frac{L\alpha}{4\xi}\right)^2\right]\right\}. \end{aligned} \tag{3.22}$$

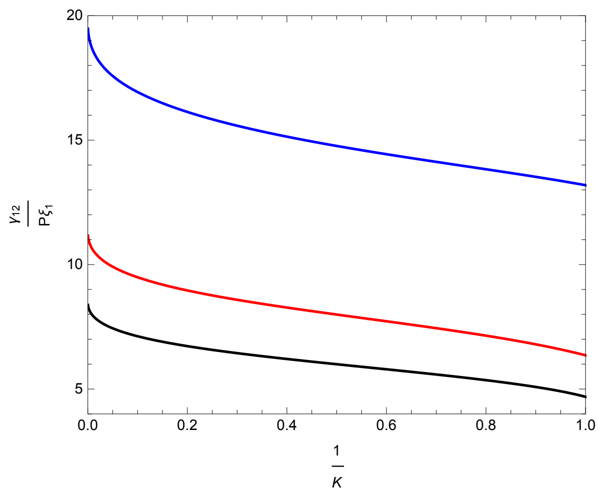

Figure 3.2: The interface tension versus $1/K$ at $L = 10$ when $\xi = 0.5$ (black line), $\xi = 1$ (red line) v $\xi = 3$ (blue line).

It is apparent that the same between the interface tension of two-component BECs and surface tension of a single BEC: when the distance between two parallel plates decreases, the interface tension decreases sharply to zero because the system is connected to the particle reservoir; when distance is large enough, the tension is constant. The different point is the role of the parameter $\xi = \xi_2/\xi_1$. When this parameter increases, the saturation amplitude of the interface tension increases.

Using the interface tension, we find the analytic equation for the Casimir-like force in the GCE

$$F_{\gamma_{12}} = -\alpha^2 e^{-\frac{L}{\alpha\xi}} \left[1 + B_2 - B_2 e^{\frac{\alpha L}{\xi}} + e^{\frac{L(1+\xi)}{\alpha\xi}} \left(1 + A_1 - 4A_1 e^{\frac{L}{\alpha}} + 3A_1 e^{\alpha L} \right) \right]. \tag{3.27}$$

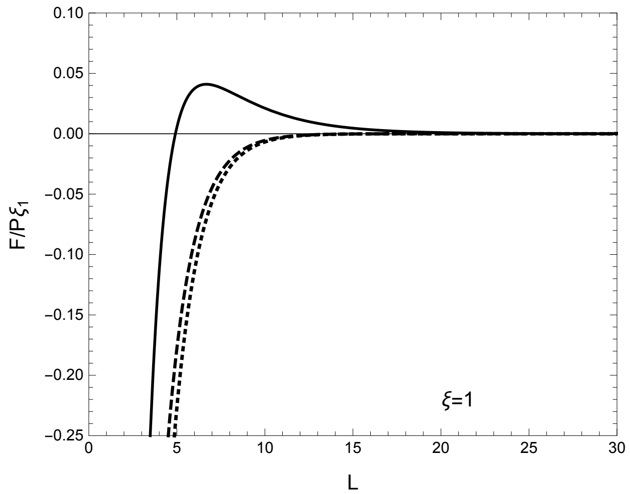

Figure 3.6: Force acts on a unit area of the walls versus L at $\xi = 1$. The solid, dashed and dotted lines correspond to $K = 1.1, K = 2$ and $K = 3$.

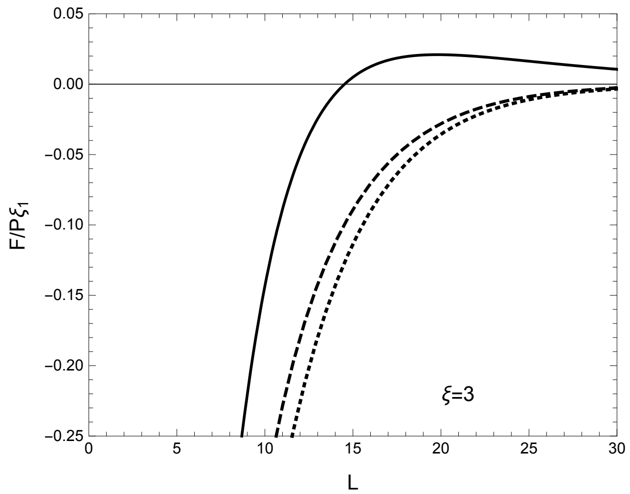

Figure 3.7: Force acts on a unit area of the walls versus L at $\xi = 3$. The soild, dashed and dotted lines correspond to $K = 1.1, K = 2$ and $K = 3$.

Based on (3.27) we plot in Fig 3.6 and Fig 3.7, it is clear that the Casimir-like force is attractive in weak segregation $K < 3$, it will become a repulsive if two plates move away from each other $L \gg \xi$. In case of strong segregation $K > 3$, Casimir-like force is always attractive and vanishing with all parameters of the system as $L \to +\infty$.

3.2. Casimir force

The same way with a single BEC, by using the quantum field theory, we study the Casimir force in two-component BEC and consider it in the one-loop and two-loop approximations.

3.2.1. In the one-loop approximation

To begin with, using the absorption term method $\triangle_{1j}\ \Omega$ to deal divergence and considering in dimensionless form, we obtain the Casimir energy

$$\mathcal{E}_{Cj} = \int_0^\infty dx \frac{\rho_j(x, \bar{L}_j)}{e^{2\pi x} - 1}, \tag{3.52}$$

where $\rho_j(x, \bar{L}_j)$ is density of state function for component j, which has the form

$$\rho_j(x, \bar{L}_j)$$
$$= \begin{cases} -\frac{g_{jj} n_{j0}}{8\pi \bar{L}_j^4 \xi_j^2} \left[x\sqrt{\bar{L}_j^2 \phi_j^2 - x^2} \left(2x^2 - \bar{L}_j^2 \phi_j^2\right) + \bar{L}_j^4 \phi_j^4 \tan^{-1}\left(\frac{x}{\sqrt{\bar{L}_j^2 \phi_j^2 - x^2}}\right)\right], \\ \qquad\qquad \text{when } 0 \le x < \bar{L}_j \phi_j; \\ -\frac{g_{jj} n_{j0} \phi_j^4}{16\xi_j^4}, \qquad \text{when } x \ge \bar{L}_j \phi_j. \end{cases} \tag{3.53}$$

Based on this, we can calculate Casimir force

$$F_C = \sum_{j=1,2} \frac{g_{jj} n_{j0}}{2\pi^2 \xi_j^2 \bar{L}_j^5} \int_0^{\bar{L}_j \phi_j} \frac{x^3 \sqrt{\bar{L}_j^2 \phi_j^2 - x^2}}{e^{2\pi x} - 1} dx. \tag{3.55}$$

It is obviously from (3.55) that Casimir force is not simple superposition of the one of two single component BEC. The interparticle interaction between two different species with strength g_{12} is embedded in ϕ_j. We can see that the Casimir force consists of the finite-size effect and the repulsive force among particles. This equation shows that:

- For ideal Bose gases $g_{jj} = 0$ the Casimir force is vanished as discussed in the works of S. Biswas for single BEC.

- In case interspecies interaction is absent $g_{12} = 0$, the system behaves as the single BEC. In this case the order parameters do not depend on g_{12}, this leads to the inverse propagators and then the thermodynamical potential also is independent of g_{12}.

- In strong segregation $g_{12} \to \infty$, two order parameters only meet and vanish at a planar interface parallel to two confined plates of the system. Because we only consider the low energy excitations, thus we can see that the attractive force caused by quantum fluctuation and repulsive force between two different species are the same order in limit of strong separation. Two opposite forces lead to the vanishing of Casimir force.

We consider now the Casimir force in large inter-distance limit. In this limit, the Casimir energy can be rewritten

$$\mathcal{E}_{Cj} = -\frac{m_j^3 c_j^2}{360\pi\hbar^2}\left(\frac{\phi_j}{\bar{L}_j^3} - \frac{1}{7\phi_j \bar{L}_j^5}\right). \tag{3.58}$$

The Casimir force can be found in this limit

$$F_C \approx \frac{1}{720\pi^2\hbar^2}\sum_{j=1,2} m_j^3 c_j^2 \left(\frac{3\phi_j}{\bar{L}_j^4} - \frac{5}{7\phi_j \bar{L}_j^6}\right). \tag{3.59}$$

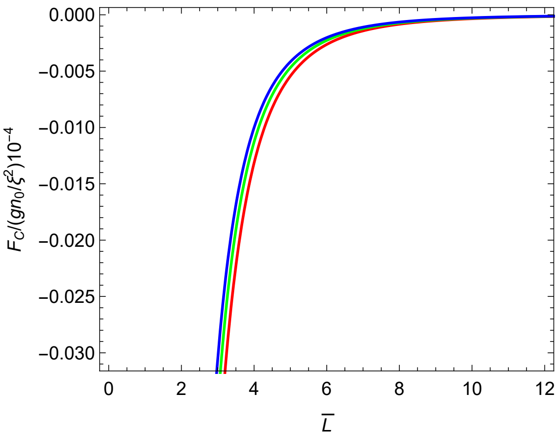

Figure 3.8: Casimir force versus $\bar{L}$ at $K = 0.5$ (red line), $K = 1$ (green line) and $K = 1.5$ (blue line).

From the graph showing evolution of Casimir force versus distance, we see that generally there is similar to that in single BEC. There is also an important result, Casimir force (and, of course, Casimir energy) will be suppressed in limit full strong segregation, an unprecedented result in previous studies. This result is checked in Fig 3.9, the Casimir force as a function of $1/K$ is plotted in (Fig 3.9a) for immiscible case and versus K for miscible case in (Fig 3.9b), we can see the suitability with Fig 3.8. It also shows that for full strong segregation $K \to \infty$ the Casimir force tends to zero.

3.2.2. In the two-loop approximation

The purpose of thesis is to investigates the finite-size effect in two-component BECs and confirms the results obtained in the one-loop approximation. We continue to study the two-component BECs in higher-order approximation (two-loop approximation) by using the CJT effective action approach.

In the same form as in Chapter 2, we obtain the gap equations and the

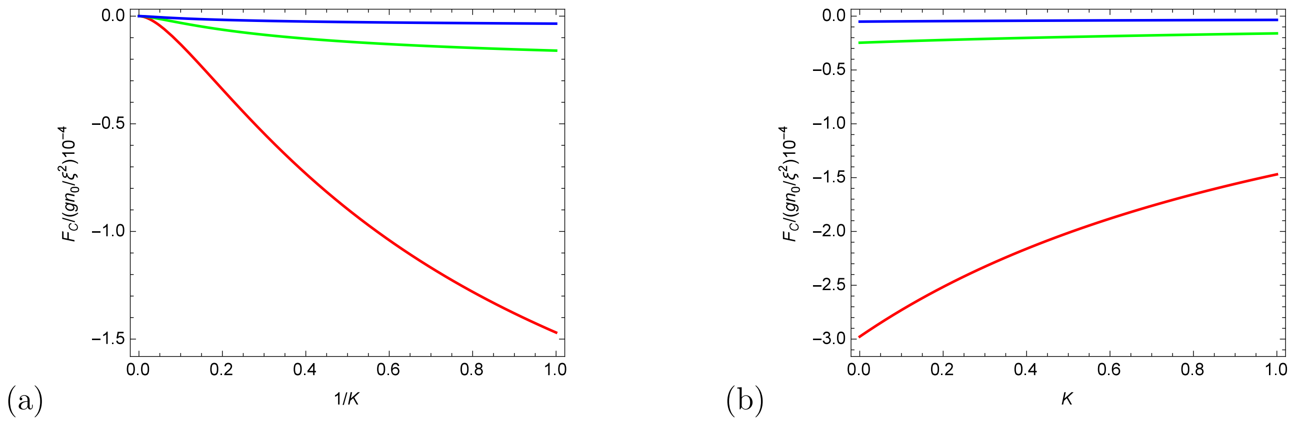

Figure 3.9: Casimir force versus the interaction parameter at $L = 1$ (red line), $L = 2$ (green line) and $L = 3$ (blue line).

SD equations describing the state of BECs in dimensionless form.

$$\begin{aligned}
-1 + \phi_1^2 + K\phi_2^2 + \frac{m_1 g_{11}\mathcal{M}_1}{24\hbar^2\ell} + K\frac{m_2 g_{22}\mathcal{M}_2}{24\hbar^2\ell} &= 0,\\
-1 + \phi_2^2 + K\phi_1^2 + \frac{m_2 g_{22}\mathcal{M}_2}{24\hbar^2\ell} + K\frac{m_1 g_{11}\mathcal{M}_1}{24\hbar^2\ell} &= 0,\\
-1 + 3\phi_1^2 + K\phi_2^2 + 3\frac{m_1 g_{11}\mathcal{M}_1}{24\hbar^2\ell} + K\frac{m_2 g_{22}\mathcal{M}_2}{24\hbar^2\ell} &= \mathcal{M}_1^2,\\
-1 + 3\phi_2^2 + K\phi_1^2 + 3\frac{m_2 g_{22}\mathcal{M}_2}{24\hbar^2\ell} + K\frac{m_1 g_{11}\mathcal{M}_1}{24\hbar^2\ell} &= \mathcal{M}_2^2.
\end{aligned} \tag{3.75}$$

Consequently, we obtain the effective mass and order parameters in dimensionless form

$$\begin{aligned}
\mathcal{M}_1^2 = \mathcal{M}_2^2 &= \frac{2}{K+1},\\
\phi_1^2 = \frac{1}{K+1} &+ \frac{g_{11}m_1}{12\hbar^2\ell\sqrt{2(K+1)}},\\
\phi_2^2 = \frac{1}{K+1} &+ \frac{g_{22}m_2}{12\hbar^2\ell\sqrt{2(K+1)}}.
\end{aligned} \tag{3.76}$$

Comparing to those in the one-loop approximation, we easily see that the compactified space has a significant effect on the order parameters, especially in region of small distance, whereas it is independent of this distance in the one-loop approximation. It is obvious that $\phi_j^2 = \phi_{j1}^2 + \Delta\phi_j^2$ and correction term

$$\Delta\phi_j^2 = \frac{m_j g_{jj}}{12\hbar^2\ell\sqrt{2(K+1)}}. \tag{3.78}$$

Equation (3.78) shows that the correction term will be vanished when distance ℓ is large enough and/or for ideal gas.

In Fig 3.10, we plot the dimensionless order parameters as functions of distance ℓ for mixtures of BEC rubidium 87 and cesium 133. The red and blue lines

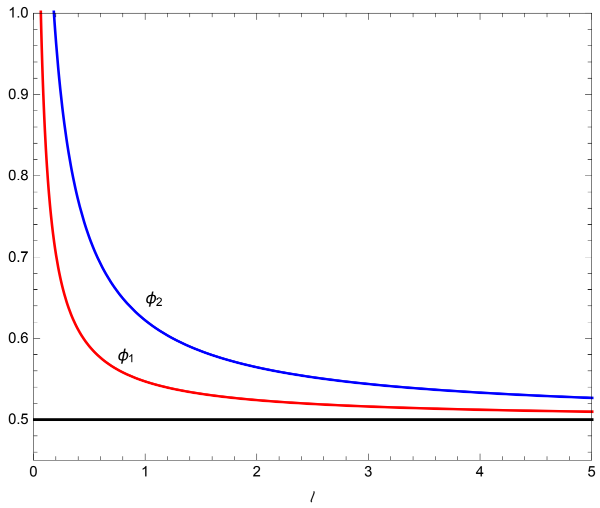

Figure 3.10: The evolution of dimensionless order parameters versus distance ℓ for rubidium (red line) and cesium (blue line)

correspond to rubidium and cesium, respectively; the black line expresses the value $1/(K+1)$, which is the value in one-loop approximation. This figure confirms the above comments and also shows different results between the IHF and the one-loop approximations.

Then, the Casimir energy per unit volume in dimensionless form has form

$$\mathcal{E}_{Cj} = -\frac{g_{jj} n_{j0} \xi_j \pi^2 \mathcal{M}_j}{1440\ell}. \tag{3.79}$$

From (3.76), we see that the effective masses in the IHF approximation are independent of the distance. We arrive the Casimir-force

$$F_C = -\frac{\hbar \pi^2}{480\ell^4 \sqrt{K+1}} \sum_{j=1,2} v_j. \tag{3.80}$$

Equation (3.80) confirms the results in our previous work that in BECs, the Casimir force is not simple superposition of the one of two single component BEC because of presence of interspecies interaction g_{12} in parameter K. In addition, this result also shows that the Casimir force is vanished in limit of strong segregation $K \to \infty$, the same as the one in the one-loop approximation.

Conslusion

In this thesis, based on the GP theory in DPA, the Cornwall-Jackiw-Tombolis effective action approach in the one-loop and two-loop approximations, we study the Casimir effect in a single BEC and two-component BECs. There are a lot of important results, which have been obtained in the framework of this thesis. The following, we review the most important results.

1. For a single BEC, when we study in CE, we find that:

   - When the distance between the two parallel plates increases, the Casimir force magnitude decreases gradually according to the half-integer power law of the distance between two hard walls;

   - There always exists a value of the distance between two parallel plates where the Casimir force is completely vanished. This occurs when the distance between the two parallel plates satisfies the equation (2.58). The obtained results, which play an important role, scientifically, and guide experimental studies on the Casimir effect in the single BEC. In technology, when we want to minimize the influence of the interaction force between the limits of an electronic component as well as the effect between the components in the same equipment, the designer and manufacturer can choose the this way. This result is also essential in the field of nanomaterials when BEC was applied like other materials in the work of H. Chan in 2001 and F. Serry in 1998.

2. In case of two-component BECs, Casimir force is vanished in the limit of the full strong segregation. The obtained results have a great significance in the research and application of BEC as mentioned above.

3. When studying the influences of finite-size effect in a single and two-component BECs, we should not ignore the contribution of higher-order diagrams in the interaction Lagrangian .

The research results of the thesis are reliable, and have been published in the prestigious scientific journals: *Journal of Statistical Physics, International Journal of Modern Physics B*, and *Journal of Experimental and Theoretical Physics.*

Published works related to the thesis

1. N. V. Thu, L. T. Theu (2017), "Casimir Force of Two-Component BoseEinstein Condensates Confined by a Parallel Plate Geometry", *Journal of Statistical Physics* **168**, 1.

2. N. V. Thu, L. T. Theu (2019), "Finite-size effect on BoseEinstein condensate mixtures in improved HartreeFock approximation", *International Journal of Modern Physics B* **33**, 1950114.

3. N. V. Thu, L. T. Theu (2018), "Influence of the Finite Size Effect on Properties of a Weakly Interacting Bose Gas in Improved Hatree-fock Approximation", *VNU Journal Of Science: Mathematics - Physics* **34**, 3.

4. L. T. Theu, N. V. Thu (2018), "Casimir force on a single interacting Bose-Einstein condensate in the Double-Parabola Approximation", *HNUE JOURNAL OF SCIENCE: Natural Sciences* **6**, 66.

5. N. V. Thu, L.T. Theu, D. T. Hai (2020), "Casimir and surface forces on a single Bose-Einstein condensate in canonical ensemble", *Jounrnal of Experimental and Theoretical Physics* **130**, 321.