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STABILITY AND REGULARITY ANALYSIS FOR SEMILINEAR RAYLEIGH-STOKES TYPE EQUATIONS

SUMMARY OF DOCTORAL THESIS IN MATHEMATICS

Major: Mathematical Analysis
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INTRODUCTION

1. Motivation and history of the problem

Consider the evolution equation:

$$\frac{\partial u}{\partial t} - \gamma \frac{\partial}{\partial t}\Delta u - \Delta u = f. \tag{1}$$

The equation (1) is the model of many real-life problems. We can call it the pseudo-parabolic equation, the nonclassical diffusion equation, the Sobolev type evolution equation, ... This equation can be the mathematical model of physics problems, for example, it was used to describe and study second-order fluids (see Coleman (1960)). This equation is the energy equation in the theory of heat conduction involving two temperatures (see Chen (1968)) and describe the dispersion of long waves (see Benjamin (1972)).

In (1), if $\frac{\partial}{\partial t}\Delta u$ is replaced by $\partial_{t,\alpha}\Delta u$, we have:

$$\frac{\partial u}{\partial t} - \gamma \partial_{t,\alpha}\Delta u - \Delta u = f, \tag{2}$$

where $\partial_{t,\alpha}$ is the Riemann-Liouville fractional derivative:

$$\partial_{t,\alpha} v(t) = \frac{d}{dt}\int_0^t \frac{1}{(t-\zeta)^\alpha \Gamma(1-\alpha)} v(\zeta) d\zeta, \alpha \in (0,1),$$

then (2) is called the Rayleigh-Stokes equation. In recent years, many problems related to Rayleigh-Stokes equation have received considerable attention, since the Rayleigh-Stokes equation plays an important role in describing the behavior of some non-Newtonian fluids. The fractional derivative $\partial_{t,\alpha}$ is used to capture the viscoelastic behavior of the flow (see Bazhlekova (2015), Fetecau (2009)).

In general, if $k \in L^1_{loc}(\mathbb{R}^+)$ is nonnegative we have the generalized Rayleigh-Stokes equation:

$$\frac{\partial u}{\partial t} - \mathfrak{D}_{t,\{k\}}\Delta u - \Delta u = f, \tag{3}$$

where $\mathfrak{D}_{t,\{k\}}$ is the nonlocal differential operator of Riemann-Liouville type defined by:

$$\mathfrak{D}_{t,\{k\}} u(t) = \frac{d}{dt}\int_0^t k(t-\zeta)\, u(\zeta)\, d\zeta,$$

here k is the kernel function. The equation (3) is generalization for many problems considered in literature. In the case k is a constant, (3) is a classical diffusion equation. If k is a regular function, e.g. $k \in C^1(\mathbb{R}^+)$ then our equation reads

$$\frac{\partial u}{\partial t} - a\Delta u - \int_0^t b(t-\zeta)\Delta u(\zeta) d\zeta = f,$$

with $a = 1 + m(0)$ and $b(t) = m'(t)$, which is a nonclassical diffusion equation. Clearly, if $k(t) = \frac{\gamma t^{-\alpha}}{\Gamma(1-\alpha)}$ then we obtain the equation (2).

For the linear Rayleigh-Stokes equation, the study of exact solution can be found in the paper Fetecau (2009), in this paper the authors consider the viscoelastic fluid to be modelled as a generalized Oldroyd-B fluid. The exact solution for the velocity field, obtained by means of the Laplace and double Fourier sine transforms, is presented under integral and series form in terms of the generalized G-functions. Other results related to exact solutions for Rayleigh-Stokes problems can be found in Shen (2006), Zierep (2007), Xue (2009) and Khan (2009).

The exact solutions for Rayleigh-Stokes problems involving infinite series and special functions, and thus are inconvenient for numerical evaluation. Hence, it is imperative to develop efficient and optimally accurate numerical algorithms for Rayleigh-Stokes problems. In the paper Bazhlekova (2015), the mathematical model is given by:

$$\frac{\partial u}{\partial t} - \Delta u - \gamma \partial_{t,\alpha} \Delta u = f \text{ in } \Sigma \subset \mathbb{R}^d, 0 < t \leq T;$$
$$u = 0 \text{ on } \partial\Sigma, 0 < t \leq T;$$
$$u(\cdot, 0) = v \text{ in } \Sigma,$$

where $\Sigma \subset \mathbb{R}^d$ is a convex polyhedral domain, $\gamma > 0$ is a constant, v is the initial data, ∂_t^α is the Riemann-Liouville fractional derivative of order $\alpha \in (0, 1)$. In this paper, for the case $f \equiv 0$, the solution is represented by the resolvent operator $S(t)$ and the regularity of the solution is studied. In addition, the authors develop a space semidiscrete Galerkin scheme using continuous piecewise linear finite elements, and derive optimal with respect to initial data regularity error estimates for the finite element approximations. Further, two fully discrete schemes based on the backward Euler method and second-order backward difference method and the related convolution quadrature are developed, and optimal error estimates are derived for the fully discrete approximations for both smooth and nonsmooth initial data. Numerical results for one-dimensional and two-dimensional examples with smooth and nonsmooth initial data are presented to illustrate the efficiency of the method, and to verify the convergence theory. For other results on numerical schemes for Rayleigh-Stokes problems, see Bi (2018), Chen (2013), Chen (2008), Salehi (2018), Zaky (2018).

For the initial data problem for nonlinear Rayleigh-Stokes equation, the existence, stability and regularity of mild solution can be found in Lan (2022), in the following model:

$$\frac{\partial u}{\partial t} - \Delta u - \gamma \partial_{t,\alpha} \Delta u = f(u) \text{ in } \Sigma, t > 0,$$
$$u = 0 \text{ on } \partial\Sigma, t \geq 0,$$
$$u(0) = \xi, \text{ in } \Sigma,$$

where $0 < \alpha < 1$, $\Sigma \subset \mathbb{R}^d$. In this paper, the author had proved the existence of mild solution. For the regularity, with suitable conditions, the mild solution is a strong one. In addition, the asymptotic stability of solution and the convergence to equilibrium point are studied. Finally, the limiting case $\alpha = 1$ is considered, in this case the resolvent operator has no smoothing effect. The qualitative results about mild solution to the initial data problem for nonlinear Rayleigh-Stokes equation can be found in Zhou (2021) and Luc (2021). The initial data problem for generalized Rayleigh-Stokes equation is presented in Ke (2022):

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial t}(k * (-\Delta)^\sigma u) - \Delta u = f(u) \text{ in } \Sigma, t > 0,$$
$$u = 0 \text{ on } \partial\Sigma, t \geq 0,$$
$$u(\cdot, 0) = \xi, \text{ in } \Sigma,$$

where $(k * v)(t) = \int_0^t k(t - \zeta)v(\zeta)d\zeta$, $(-\Delta)^\sigma$ denotes the fractional power of the Laplacian, $\sigma \in [0, 1]$, $\xi \in L^2(\Sigma)$. When $\sigma = 1$ and $k(t) = \dfrac{m_0 t^{-\alpha}}{\Gamma(1-\alpha)}, m_0 > 0$, $0 < \alpha < 1$, we obtain the Rayleigh-Stokes equation. The authors's aim is to analyze some sufficient conditions ensuring the global solvability, regularity and stability of solution. In particular, they study the existence of mild solution, and thanks to suitable conditions the mild solution can become strong solution. Moreover, they prove some stability results such as the dissipativity, the asymptotic stability and the convergence to equilibrium.

In the theory of nonlocal partial differential equation, inverse problem received considerable attention. When consider the inverse problem for Rayleigh-Stokes equation, we want to study the identification problem and the final value problem. In the identification problem, the source term is not fully obtained from observation, and a nonlinear perturbation may involve. Therefore we need a measurement at final time. In the case of linear Rayleigh-Stokes equation, the paper Triet (2018) has the following model:

$$\begin{aligned} \frac{\partial u}{\partial t} - \Delta u - \gamma \partial_{t,\alpha} \Delta u &= F(x,t), (x,t) \in \Sigma \times (0,T), \\ u(x,t) &= 0, x \in \partial\Sigma, \\ u(x,0) &= u_0(x), x \in \Sigma, \end{aligned}$$

where $F(x,t)$ is unknown, therefore we need observation data involving "noise". In this paper, to regularize the unstable solution, the authors apply a general filter method for constructing regularized solution, and the convergence rate of this method also has been investigated.

In the final value problem, the observation of initial state is unavailable, and we make use of the observation at present time to detect the previous states of the system. The final value problem for (2) can be found in Luc (2019):

$$\begin{aligned} \frac{\partial u}{\partial t} - \Delta u - \gamma \partial_{t,\alpha} \Delta u &= F(x,t), (x,t) \in \Sigma \times (0,T), \\ u(x,t) &= 0, x \in \partial\Sigma, \\ u(x,T) &= f(x), x \in \Sigma, \end{aligned}$$

where $\Sigma \subset \mathbb{R}^d$. The main results of this paper are finding the representation of solution to the final value problem and showing that this solution is Hölder regular. Other results when the right-hand side of the Rayleigh-Stokes equation contains the state function u (for example $G(x,t,u)$) can be found in Ngoc (2021), Tuan (2019).

The first problem we propose is about finding the suitable conditions such that the mild solution exists and analyzing the stability, the regularity of the mild solution with respect to the nonlinear equation (2).

Precisely, we consider the model in Lan (2022) in the case the nonlinear function in the right-hand side contains a delayed term. We have:

$$\begin{aligned} \frac{\partial u}{\partial t} - \Delta u - \gamma \partial_{t,\alpha} \Delta u &= f(t, u_\theta) \text{ in } \Sigma \subset \mathbb{R}^d, t > 0, \\ u &= 0 \text{ on } \partial\Sigma, t \geq 0, \\ u(x,\zeta) &= \phi(x,\zeta), x \in \Sigma, \zeta \in [-\tau, 0], \end{aligned}$$

where $\gamma > 0$, $\alpha \in (0,1)$. In this model, u_θ is given by $u_\theta(x,t) = u(x, t-\theta(t))$, here θ is continuous on $(0,+\infty)$ and satisfies that $-\tau \leq t - \theta(t) \leq t$ and $t - \theta(t) \to +\infty$ as $t \to +\infty$, $f : (0,+\infty) \times L^2(\Sigma) \to L^2(\Sigma)$ is a nonlinear map and $\phi \in C([-\tau,0]; L^2(\Sigma))$ is given. The nonlinearity f contains a delayed term and represents the external force, which depends on the history state of the system. For example, $\theta(t) = (1-a)t + \tau$, then $u_\theta(x,t) = u(x, at - \tau)$ with $a \in (0,1]$, which is a proportional delay.

In partial differential equation, it is natural to consider the delayed terms when we want to describe real-life processes. According to our knowledge, there was no result on the stability with respect to the equation (2) involving delays, which has been pulished. Therefore, we want to study the solvability and stability of this problem. Precisely, our aims are

- analyzing the existence of mild solution when f has a superlinear growth or sublinear growth.
- analyzing the dissipativity of the system and the asymptotic stability of zero solution.
- analyzing the existence of a compact set of decay solutions.

For the equation (3), our researches focus on the identification problem and the final value problem.

If the right-hand side of (3) is replaced by $a(x)b(t)+g(u)$, where $g(u)$ is a nonlinear perturbation and $a(x)b(t)$ is an external force written as separation of variables, in that the term $a(x)$ is unknown then we have the following identification problem:

$$\begin{aligned} \frac{\partial u}{\partial t} - \Delta u - \mathfrak{D}_{t,\{k\}}\Delta u &= a(x)b(t) + g(u) \text{ in } \Sigma, t \in (0,T), \\ u &= 0 \text{ on } \partial\Sigma, t \geq 0, \\ u(0) &= \phi \text{ in } \Sigma, \\ u(T) &= \xi(u) \text{ in } \Sigma, \end{aligned}$$

where $u(t)$ takes values in $L^2(\Sigma)$, b, g, ξ and ϕ are given functions.

This problem is in the situation that the source term $a(x)b(t)$ containing $a(x)$, which is not obtained from observations. In addition, a nonlinear perturbation g may involve. In order to identify a, we need a measurement at final time, e.g. $u(T) = \xi(u)$, in this case the measured data is possibly implicit. According to our knowledge, there was no result on identifying the source term for the Rayleigh-Stokes problem involving superlinear perturbations which has been published. For this problem, we want to

- find the conditions such that the mild solution exists and study the stability of this solution.
- find the conditions such that the mild solution becomes a strong solution.

The problem of identifying source term is carried out in the case that a nonlinear perturbation involves and the final measurement is implicit. This is a practical situation which results in significant difficulties for our analysis.

The final value problem for (3) has the form

$$\begin{aligned} \frac{\partial u}{\partial t} - \Delta u - \mathfrak{D}_{t,\{k\}}\Delta u &= g(u) \text{ in } \Sigma, t \in (0,T), \\ u &= 0 \text{ on } \partial\Sigma, t > 0, \\ u(T,\cdot) &= \xi \text{ in } \Sigma, \end{aligned}$$

where u is the unkown function defined on $(0,T] \times \Sigma$, g is a given function. The final value condition $u(T,\cdot) = \xi$ is usually imposed when the observation of initial state is unavailable.

The final value problem was considered for nonlinear Rayleigh-Stokes equation in Ngoc (2021), Tuan (2019), where $g = g(u)$ and takes values in $L^2(\Sigma)$. Nevertheless, this setting does not allow $g(u)$ to admit a polynomial form, such as $g(u) = |u|^q$ with $q > 1$, and it is not likely for g to contain a gradient term. We deal with this issue by assuming that $g(u)$ belongs to $\mathbb{H}^{-\theta}$ with $\theta > 0$ (the spaces $\mathbb{H}^{\varrho}$, $\varrho \in \mathbb{R}$ are Hilbert scales). In this work, we aim at

- finding reasonable conditions on g and all parameters involved for ensuring the solvability of the problem.
- finding reasonable conditions such that the obtained solution is Hölder continuous with respect to the time variable.

From the technical point of view, since $g(u)$ takes values in dual spaces, one has no direct estimates for the Hölder continuity of solutions.

In conclusion, we have studied and completed the thesis: "*Stability and regularity analysis for semilinear Rayleigh-Stokes type equations*".

2. Purpose, objects and scope of the thesis

2.1. Purpose

In the thesis, we study the mild solution to forward and inverse problem for (2) and (3), respectively. More specifically, we study the existence of mild solution and its qualitative properties.

2.2. Objects

Problem proposed for Rayleigh-Stokes equation and generalized Rayleigh-Stokes equation.

2.3. Scope

The scope of the thesis includes:
- The existence and the stability of mild solution to the Cauchy problem for Rayleigh-Stokes equation involving delays.
- The solvability, stability and regularity of mild solution to the identification problem for the generalized Rayleigh-Stokes equation with a measurement at final time.
- The existence and uniqueness, and the Hölder continuity with respect to the time variable of mild solution to the final value problem for the generalized Rayleigh-Stokes equation.

3. Research methods

In this thesis, we employ the theory of completely positive functions, the theory of resolvents, Grönwall and Halanay type inequalities, Hilbert scales, fractional Sobolev spaces, fixed-point theorems.

4. Structure of thesis

The main content of the thesis is divided into 4 chapters:
- Chapter 1: *Preliminaries.*
- Chapter 2: *The Cauchy problem for Rayleigh-Stokes equation involving delays.*
- Chapter 3: *An identification problem for generalized Rayleigh-Stokes equation involving superlinear pertubations.*
- Chapter 4: *Final value problem for generalized Rayleigh-Stokes type equations involving weak-valued nonlinearities.*

5. Thesis significance

The contribution of the thesis is the new results on the solvability, stability and regularity of mild solution in nonlocal partial differential equations in general, and the nonlinear Rayleigh-Stokes equation in particular. The main results of the thesis have been published in 03 prestige international journals (we list in the section "Author's works related to the thesis that have been published"). Results of the thesis have been reported at:
1) Seminar of Department of Analysis, Faculty of Mathematics, Hanoi Pedagogical University 2;
2) 10^{th} Vietnam Mathematical Congress, 2023.

Chapter 1

PRELIMINARIES

1.1. Function spaces

In this section, we recall the notions of Laplace operators, spectrum of Laplace operators, fractional Laplace operators, Hilbert scales and fractional Sobolev spaces.

1.2. Laplace transform and fractional calculus

In this section, we recall the concepts and properties related to Laplace transform, Laplace inverse transform, Laplace transform of convolution. In addition, we present the concept of fractional integral and fractional derivative, the formula for the Laplace transform of a fractional derivative of order α.

1.3. The resolvent function and the resolvent operator

This section is devoted to present the theory of completely positive functions, the theory of resolvent functions and resolvent operators, and the representation for solutions given by resolvent operators.

In order to find the representation for solution for the inverse problem, we first find the solution representation for the initial data problem. We consider the following relaxation problem:

$$r'(t) + \lambda r(t) + \lambda \mathfrak{D}_{t,\{k\}} r(t) = 0, \ t > 0, \tag{1.1}$$

$$r(0) = 1, \tag{1.2}$$

where the unknown r is a scalar function, $\lambda > 0$.

We find a representation for solution of the linear initial value problem:

$$\frac{\partial u}{\partial t} - \Delta u - \mathfrak{D}_{t,\{k\}} \Delta u = V \ \text{ in } \Sigma, \ \text{when } t \in (0,T], \tag{1.3}$$

$$u = 0 \ \text{ on } \partial\Sigma, \ \text{when } t \in [0,T], \tag{1.4}$$

$$u(0) = \phi \ \text{ in } \Sigma, \tag{1.5}$$

where $V \in C([0,T]; L^2(\Sigma))$.

Let $\{e_i\}_{i=1}^{+\infty}$ be an orthonormal basis of $L^2(\Sigma)$ consisting of eigenfunctions of $-\Delta$ subject to the homogeneous boundary condition, i.e.,

$$-\Delta e_i = \lambda_i e_i \text{ in } \Sigma, \ e_i = 0 \text{ on } \partial\Sigma,$$

where one can assume that $0 < \lambda_1 \leq \lambda_2 \leq \ldots, \lambda_i \to +\infty$ as $i \to +\infty$.

Then we get

$$u(t) = \sum_{i=1}^{+\infty} u_i(t)e_i,\ V(t) = \sum_{i=1}^{+\infty} V_i(t)e_i.$$

From (1.3) we have

$$\begin{aligned} &u_i'(t) + \lambda_i \left(1 + \mathfrak{D}_{t,\{k\}}\right) u_i(t) = V_i(t), \\ &u_i(0) = \phi_i := \langle \phi, e_i \rangle. \end{aligned}$$

This implies

$$u(t) = \mathfrak{R}(t)\phi + \int_0^t \mathfrak{R}(t-\zeta)V(\zeta)d\zeta, \tag{1.6}$$

where $\mathfrak{R}(t)$ defined by

$$\mathfrak{R}(t)\phi = \sum_{i=1}^{+\infty} r(t,\lambda_i)\phi_i e_i,\ \phi \in L^2(\Sigma). \tag{1.7}$$

We call $r(t,\lambda)$ *the resolvent function*, and $\mathfrak{R}(t)$ *the resolvent operator*.

1.4. Fixed-point theorem

In this section, we recall the Banach fixed-point theorem, the Schauder fixed-point theorem, and the fixed-point theorem for condensing map.

Chapter 2

THE CAUCHY PROBLEM FOR RAYLEIGH-STOKES EQUATION INVOLVING DELAYS

Our aim in this chapter is to analyze some circumstances, in those the global solvability, and asymptotic behavior of solutions are addressed. By establishing a Halanay type inequality, we show the dissipativity and asymptotic stability of solutions to our problem. In addition, we prove the existence of a compact set of decay solutions by using local estimates and fixed-point arguments. The content of this chapter is based on the paper [1] in the author's works related to the thesis that have been published.

2.1. Problem setting

Let $\Sigma \subset \mathbb{R}^d$ be a bounded domain with smooth boundary $\partial\Sigma$. Consider the problem:

$$\frac{\partial u}{\partial t} - \Delta u - \gamma \partial_{t,\alpha} \Delta u = f(t, u_\theta) \text{ in } \Sigma \text{ when } t > 0, \tag{2.1}$$

$$u = 0 \text{ on } \partial\Sigma \text{ when } t \geq 0, \tag{2.2}$$

$$u(x, \zeta) = \phi(x, \zeta) \text{ when } x \in \Sigma, \zeta \in [-\tau, 0], \tag{2.3}$$

where $0 < \alpha < 1$, $\gamma > 0$, $\partial_{t,\alpha}$ stands for the Riemann-Liouville derivative of order α, given by $\partial_{t,\alpha} v(t) = \frac{d}{dt} \int_0^t \frac{1}{(t-\zeta)^\alpha \Gamma(1-\alpha)} v(\zeta) d\zeta$, for $t > 0$; $f : (0, +\infty) \times L^2(\Sigma) \to L^2(\Sigma)$ is a nonlinear map and $\phi \in C([-\tau, 0]; L^2(\Sigma))$ is given.

The delayed term u_θ is defined by $u_\theta(x,t) = u(x, t-\theta(t))$ with θ being a continuous function on $(0, +\infty)$ such that $-\tau \leq t - \theta(t) \leq t$ $\forall t \in (0, +\infty)$, and $t - \theta(t) \to +\infty$ as $t \to +\infty$.

2.2. Solution formula

In (1.1) and (1.3), when $k(t) = \frac{\gamma t^{-\alpha}}{\Gamma(1-\alpha)}$ we obtain the respective $r(t, \lambda)$ and $\mathfrak{R}(t)$. Properties of $\mathfrak{R}(t)$ are given in the following lemma.

Lemma 2.1. *For any $T > 0$, $\forall \nu \in L^2(\Sigma)$ we have:*

(a) $\mathfrak{R}(\cdot)\nu \in C([0, T]; L^2(\Sigma))$ and $\mathfrak{R}(\cdot)\nu \in C((0, T]; \mathfrak{H}^2(\Sigma) \cap \mathfrak{H}_0^1(\Sigma))$.

(b) $\|\mathfrak{R}(t)\nu\| \leq r(t, \lambda_1)\|\nu\|$ and $\|\mathfrak{R}(t)\| \leq 1$ for all $t \geq 0$.

(c) $\mathfrak{R}(\cdot)\nu \in C^{(n)}((0,T];L^2(\Sigma))$ *for all* $n \in \mathbb{Z}, n \geq 0$*, and* $\|\mathfrak{R}^{(n)}(t)\nu\| \leq \frac{C}{t^n}\|\nu\|$*, where* $C > 0$ *is a constant.*

(d) $\|\Delta\mathfrak{R}^{(n)}(t)\nu\| \leq \frac{C}{t^{n+1-\alpha}}\|\nu\|$ *for all* $t > 0$ *and* $n \in \mathbb{Z}, n \geq 0$.

The next lemma is about the Halanay type inequality, which we use to study the stability of the mild solution:

Lemma 2.2. *Let* ν *be continuous and nonnegative function satisfying*

$$\nu(t) \leq r(t,\lambda)\nu_0 + \int_0^t r(t-\zeta,\lambda)[d_1 \sup_{\eta\in[\zeta-\theta(\zeta),\zeta]} \nu(\eta) + d_2(\zeta)]d\zeta \text{ when } t > 0, \tag{2.4}$$

and

$$\nu(\zeta) = \psi(\zeta) \text{ with } \zeta \in [-\tau, 0], \tag{2.5}$$

where $0 < d_1 < \lambda$, $\psi \in C([-\tau,0];\mathbb{R}^+)$ *and* $d_2 \in L^1_{loc}(\mathbb{R}^+)$ *is a nondecreasing function. Then*

$$\nu(t) \leq \frac{\lambda}{\lambda - d_1}\Big[\nu_0 + \int_0^t r(t-\zeta,\lambda)d_2(\zeta)d\zeta\Big] + \sup_{\zeta\in[-\tau,0]} \psi(\zeta) \text{ for all } t > 0. \tag{2.6}$$

In addition, if $r(\cdot,\lambda) * d_2$ *is bounded on* $(0,+\infty)$ *then*

$$\limsup_{t\to+\infty} \nu(t) \leq \sup_{(0,+\infty)} \int_0^t r(t-\zeta,\lambda)d_2(\zeta)d\zeta. \tag{2.7}$$

In particular, if $d_2 = 0$ *then* $\nu(t) \to 0$ *as* $t \to +\infty$.

2.3. Solvability and stability

We give the definition of the mild solution as follows:

Definition 2.1. Let $\phi \in C([-\tau,0];L^2(\Sigma))$ be given. A function $u \in C([-\tau,T];L^2(\Sigma))$ is said to be a mild solution to (2.1)-(2.3) on $[-\tau,T]$ iff $u(\cdot,\zeta) = \phi(\cdot,\zeta)$ for $\zeta \in [-\tau,0]$ and $u(\cdot,t) = \mathfrak{R}(t)\phi(\cdot,0) + \int_0^t \mathfrak{R}(t-\zeta)f(\zeta,u_\theta(\cdot,\zeta))d\zeta,\ t \in [0,T]$.

The solvability of (2.1)-(2.3) will be show in the next theorems.

Theorem 2.1. *Let* $f : [0,T] \times L^2(\Sigma) \to L^2(\Sigma)$ *be a continuous mapping such that*

(**G**1) $\|f(t,\nu)\| \leq \ell(t)\mathcal{O}(\|\nu\|)$ *for* $0 \leq t \leq T$ *and* $\nu \in L^2(\Sigma)$*, where* $\ell \in L^1(0,T)$ *is a nonnegative function and* $\mathcal{O}$ *is a continuous and nonnegative function obeying that* $\limsup_{s\to0} \frac{\mathcal{O}(s)}{s} \cdot \sup_{0\leq t\leq T} \int_0^t r(t-\zeta,\lambda_1)\ell(\zeta)d\zeta < 1$.

Then there exists $\delta > 0$ *such that the problem* (2.1)-(2.3) *has at least one mild solution on* $[-\tau,T]$*, provided that* $\|\phi\|_\infty \leq \delta$.

Theorem 2.2. *Let* $f : [0,T] \times L^2(\Sigma) \to L^2(\Sigma)$ *be a continuous mapping such that*

(**G**2) $\|f(t,\nu)\| \leq \ell(t)(1 + \|\nu\|)$ *for* $0 \leq t \leq T$ *and* $\nu \in L^2(\Sigma)$*, where* $\ell \in L^1(0,T)$ *is a nonnegative function.*

Then the problem (2.1)-(2.3) *has at least one mild solution on* $[-\tau,T]$.

Theorem 2.3. *Let* $f : [0,T] \times L^2(\Sigma) \to L^2(\Sigma)$ *be a continuous mapping such that*

(**G**3) $f(\cdot,0) = 0$ *and* $\|f(t,\nu_1) - f(t,\nu_2)\| \leq \ell(t)\mathcal{K}(s)\|\nu_1 - \nu_2\|$ *for* $0 \leq t \leq T$ *and* $\nu_1,\nu_2 \in L^2(\Sigma)$ *such that* $\|\nu_1\|,\|\nu_2\| \leq s$*, where* $\ell \in L^1(0,T)$ *is a nonnegative function and* $\mathcal{K}$ *is a continuous function obeying that*

$$\limsup_{s\to0} \mathcal{K}(s) \cdot \sup_{0\leq t\leq T} \int_0^t r(t-\zeta,\lambda_1)\ell(\zeta)d\zeta < 1.$$

Then there exists $\delta > 0$ such that the problem (2.1)-(2.3) *has a unique mild solution on* $[-\tau, T]$*, provided that* $\|\phi\|_\infty \le \delta$.

We are now in a position to show the dissipativity of our system.

Theorem 2.4. *For all $T > 0$, assume that all the hypotheses of Theorem 2.2 hold. Assume that $\|\ell\|_{\text{ess sup}} := \text{ess sup}_{t\ge 0}\ell(t) < \lambda_1$. Then there exists a bounded absorbing set for solution of* (2.1)-(2.3)*, with arbitrary initial data.*

The next theorem shows the asymptotic stability of zero solution to (2.1)-(2.2).

Theorem 2.5. *Let $f : \mathbb{R}^+ \times L^2(\Sigma) \to L^2(\Sigma)$ be a continuous mapping such that:*

(**G**4) *$f(\cdot, 0) = 0$ and $\|f(t, \nu_1) - f(t, \nu_2)\| \le \ell(t)\mathcal{K}(s)\|\nu_1 - \nu_2\|$ for $0 < t < +\infty$ and $\nu_1, \nu_2 \in L^2(\Sigma)$ such that $\|\nu_1\|, \|\nu_2\| \le s$, where $\ell \in L^\infty(\mathbb{R}^+)$ is a nonnegative function and $\mathcal{K}$ is a continuous function satisfying that*

$$\|\ell\|_{\text{ess sup}} \cdot \limsup_{s\to 0} \mathcal{K}(s) < \lambda_1.$$

Then the zero solution of (2.1)-(2.2) *is asymptotically stable.*

2.4. Existence of decay solutions

Assume that

(**G**5) $f : (0, +\infty) \times L^2(\Sigma) \to L^2(\Sigma)$ is a continuous mapping such that

$$\|f(t, \nu)\| \le \ell(t)\mathcal{O}(\|\nu\|),\ \forall t \in (0, +\infty), \nu \in L^2(\Sigma),$$

where $\ell \in L^1_{loc}(\mathbb{R}^+)$ is a nonnegative function and $\mathcal{O} \in C(\mathbb{R}^+)$ is a nonnegative and nondecreasing function such that

$$\limsup_{s\to 0} \frac{\mathcal{O}(s)}{s} \cdot \sup_{t\ge 0} \int_0^t r(t-\zeta, \lambda_1)\ell(\zeta)d\zeta < 1, \tag{2.8}$$

and

$$\lim_{T\to+\infty} \sup_{t\ge T} \int_0^{\frac{t}{2}} r(t-\zeta, \lambda_1)\ell(\zeta)d\zeta = 0. \tag{2.9}$$

The next theorem represents the main result of this section.

Theorem 2.6. *Let* (**G**5) *hold. Then there exists $\delta > 0$ such that the problem* (2.1)-(2.3) *has a compact set of decay solutions, provided that $\|\phi\|_\infty \le \delta$.*

2.5. Examples for (2.8) and (2.9)

Firstly, we consider (2.9). Let $\ell \in L^\infty(\mathbb{R}^+)$ and $\|\ell\|_{\text{ess sup}} = \text{ess sup}_{t\ge 0}|\ell(t)|$. Then (2.9) holds. Indeed, we can see that

$$\begin{aligned}\sup_{t\ge T} \int_0^{\frac{t}{2}} r(t-\zeta, \lambda_1)\ell(\zeta)d\zeta &\le \|\ell\|_{\text{ess sup}} \sup_{t\ge T} \int_0^{\frac{t}{2}} r(t-\zeta, \lambda_1)d\zeta \\ &\le \|\ell\|_{\text{ess sup}} \sup_{t\ge T} \int_{\frac{t}{2}}^{t} r(\zeta, \lambda_1)d\zeta \\ &\le \|\ell\|_{\text{ess sup}} \int_{\frac{T}{2}}^{+\infty} r(\zeta, \lambda_1)d\zeta \to 0 \text{ as } T \to +\infty,\end{aligned}$$

thanks to $r(\cdot,\lambda_1)\in L^1(\mathbb{R}^+)$.

Secondly, we consider (2.8). If f has superlinear growth, for example $\mathcal{O}(s)=s^p$ with $p>1$, then (2.8) holds. If f has a sublinear growth, for example $\mathcal{O}(s)=s$, then (2.8) becomes

$$\sup_{t\geq 0}\int_0^t r(t-\zeta,\lambda_1)\ell(\zeta)d\zeta<1. \tag{2.10}$$

Noting that

$$\int_0^t r(t-\zeta,\lambda_1)\ell(\zeta)d\zeta\leq\|\ell\|_{\text{ess sup}}\int_0^t r(\zeta,\lambda_1)d\zeta\leq\|\ell\|_{\text{ess sup}}\lambda_1^{-1},$$

we get that (2.10) is fulfilled, provided that $\|\ell\|_{\text{ess sup}}<\lambda_1$.

Chapter 3

AN IDENTIFICATION PROBLEM FOR GENERALIZED RAYLEIGH-STOKES EQUATIONS INVOLVING SUPERLINEAR PERTUBATIONS

In this chapter, we prove the unique solvability and stability of solution. Furthermore, we show that the obtained solution is differentiable and it is a strong one. The content of this chapter is based on the paper [2] in the author's works related to the thesis that have been published.

3.1. Problem setting

Let Σ be a bounded domain in $\mathbb{R}^d$ with smooth boundary $\partial\Sigma$. Consider the following problem: find (a, u) satisfying that

$$\frac{\partial u}{\partial t} - \Delta u - \mathfrak{D}_{t,\{k\}}\Delta u = a(x)b(t) + g(u) \text{ in } \Sigma \text{ when } t \in (0, T), \tag{3.1}$$

$$u = 0 \text{ on } \partial\Sigma \text{ when } t \geq 0, \tag{3.2}$$

$$u(0) = \phi \text{ in } \Sigma, \tag{3.3}$$

$$u(T) = \xi(u) \text{ in } \Sigma, \tag{3.4}$$

where $u(\cdot)$ takes values in $L^2(\Sigma)$, b, g and ξ are given functions, $\mathfrak{D}_{t,\{k\}}$ stands for the nonlocal differential operator of Riemann-Liouville type defined by

$$\mathfrak{D}_{t,\{k\}}\nu(t) = \frac{d}{dt}\int_0^t k(t-\zeta)\nu(\zeta)d\zeta,$$

with respect to the kernel function $k \in L^1_{loc}(\mathbb{R}^+)$. In this problem, we need the following conditions:

(M) *$k \in L^1_{loc}(\mathbb{R}^+)$ is nonnegative and $\mathfrak{a}(t) = 1 + k(t)$ is completely positive.*

(NND) *There exists a positive nonincreasing function $m \in L^1_{loc}(\mathbb{R}^+)$ such that*

$$m * \mathfrak{a} = 1 \text{ on } \mathbb{R}^+.$$

3.2. Estimations for resolvent operator and representation of the mild solution

We consider the resolvent operator $\mathfrak{R}(t)$ as in (1.7). The following lemma give us the properties of $\mathfrak{R}(t)$:

Lemma 3.1. *Let $\{\mathfrak{R}(t)\}_{t\geq 0}$ be the resolvent operator defined by* (1.7), $\nu \in L^2(\Sigma)$ *and* $T > 0$. *Then*

(a) $\mathfrak{R}(\cdot)\nu \in C([0,T]; L^2(\Sigma))$ *and* $\|\mathfrak{R}(t)\| \leq r(t, \lambda_1)$ $\forall t \geq 0$.

(b) $\Delta\mathfrak{R}(\cdot)\nu \in C((0,T]; L^2(\Sigma))$, $\Delta\mathfrak{R}(\cdot)\nu \in L^1(0,T; L^2(\Sigma))$ *and* $\|\Delta\mathfrak{R}(t)\| \leq t^{-1}$ *for all* $t > 0$. *Moreover,*

$$\| \int_0^t \Delta\mathfrak{R}(\zeta)\nu d\zeta\| \leq \|\nu\| \ \forall t \geq 0. \tag{3.5}$$

(c) If k is nonincreasing then $\mathfrak{R}(\cdot)\nu \in C^1((0,T]; L^2(\Sigma))$ *and it holds that*

$$\|\mathfrak{R}'(t)\| \leq t^{-1} \ \forall t > 0. \tag{3.6}$$

(d) If $h \in C([0,T]; L^2(\Sigma))$ *then* $(-\Delta)^{\frac{1}{2}}\mathfrak{R} * h \in C([0,T]; L^2(\Sigma))$ *and it holds that*

$$\|(-\Delta)^{\frac{1}{2}}\mathfrak{R} * h(t)\| \leq \left(\int_0^t r(t-\zeta, \lambda_1)\|h(\zeta)\|^2 d\zeta\right)^{\frac{1}{2}}, \ \forall t \geq 0.$$

From the formula (1.6) with respect to (1.3)-(1.5), we formulate a representation of solution to (3.1)-(3.4). Let u be a solution to (3.1)-(3.4). Put $\xi_i = \langle \xi(u), e_i\rangle$, $g_i(t) = \langle g(u(t)), e_i\rangle$ (the scalar product on $L^2(\Sigma)$), and $a_i = \langle a, e_i\rangle_{\mathbb{H}^{-1},\mathbb{H}^1}$ (the duality pairing $\langle\cdot,\cdot\rangle_{\mathbb{H}^{-\varrho},\mathbb{H}^{\varrho}}$ on $\mathbb{H}^{-\varrho} \times \mathbb{H}^{\varrho}$), then

$$u(t) = \sum_{i=1}^{+\infty}\Big[r(t,\lambda_i)\phi_i + \int_0^t r(t-\zeta,\lambda_i)a_i b(\zeta)d\zeta + \int_0^t r(t-\zeta,\lambda_i)g_i(\zeta)d\zeta\Big]e_i.$$

It follows that

$$\xi_i = r(T,\lambda_i)\phi_i + \int_0^T r(T-\zeta,\lambda_i)a_i b(\zeta)d\zeta + \int_0^T r(T-\zeta,\lambda_i)g_i(\zeta)d\zeta.$$

So

$$a_i = \left(\int_0^T r(T-\zeta,\lambda_i)b(\zeta)d\zeta\right)^{-1}\Big[\xi_i - r(T,\lambda_i)\phi_i - \int_0^T r(T-\zeta,\lambda_i)g_i(\zeta)d\zeta\Big],$$

provided that $\left(\int_0^T r(T-\zeta,\lambda_i)b(\zeta)d\zeta\right)^{-1}$ is definite. Therefore, we need the following assumption:

The real-valued function b defined on $[0,T]$ is continuous on $[0,T]$, nonnegative and satisfies that $b_T := \int_0^T b(\zeta)d\zeta > 0$.

Consider the operator $\mathcal{S}$:

$$\mathcal{S}\nu = \sum_{i=1}^{+\infty}\left(\int_0^T r(T-\zeta,\lambda_i)b(\zeta)d\zeta\right)^{-1}\nu_i e_i,$$

with the domain

$$D(\mathcal{S}) = \{\nu \in L^2(\Sigma) : \sum_{i=1}^{+\infty}\left(\int_0^T r(T-\zeta,\lambda_i)b(\zeta)d\zeta\right)^{-2}\nu_i^2 < +\infty\}.$$

Then

$$a = \mathcal{S}\big[\xi(u) - \mathfrak{R}(T)\phi - \int_0^T \mathfrak{R}(T-\zeta)g(u(\zeta))d\zeta\big],$$

$$u(t) = \mathfrak{R}(t)\phi + \int_0^t \mathfrak{R}(t-\zeta)ab(\zeta)d\zeta + \int_0^t \mathfrak{R}(t-\zeta)g(u(\zeta))d\zeta.$$

Lemma 3.2. *With $b_T = \int_0^T b(\zeta)d\zeta > 0$, we have the following properties of $\mathcal{S}$:*

(a) $D(\mathcal{S}) = D(-\Delta)$.

(b) If $\nu \in \mathbb{H}^2$ then $\mathcal{S}\nu \in L^2(\Sigma)$ and $\|\mathcal{S}\nu\| \le b_T^{-1}\|\nu\| + b_T^{-1}m(T)^{-1}\|(-\Delta)\nu\|$, here m is given as in (NND).

(c) If $\nu \in \mathbb{H}^1$ then $\mathcal{S}\nu \in \mathbb{H}^{-1}$ and

$$\|\mathcal{S}\nu\|_{\mathbb{H}^{-1}} \le b_T^{-1}\lambda_1^{-\frac{1}{2}}\|\nu\| + b_T^{-1}m(T)^{-1}\|(-\Delta)^{\frac{1}{2}}\nu\|.$$

(d) If $g \in C([0,T];L^2(\Sigma))$ and $\|g\|_\infty = \sup_{[0,T]}\|g(t)\|$ then $\mathcal{S}(\mathfrak{R} * g)(T) \in \mathbb{H}^{-1}$ and

$$\|\mathcal{S}(\mathfrak{R} * g)(T)\|_{\mathbb{H}^{-1}} \le b_T^{-1}\lambda_1^{-\frac{1}{2}}[\lambda_1^{-1} + m(T)^{-1}]\|g\|_\infty. \tag{3.7}$$

(e) Assume that $g \in C^{0,\iota}((0,T];L^2(\Sigma))$ for $0 < \iota < 1$, this means there exists $C_g > 0$ such that for all $t > 0$, $\epsilon \in (0, T-t)$ we have

$$\|g(t+\epsilon) - g(t)\| \le C_g\epsilon^\iota.$$

Then $\mathcal{S}(\mathfrak{R} * g)(T) \in L^2(\Sigma)$ and

$$\|\mathcal{S}(\mathfrak{R} * g)(T)\| \le b_T^{-1}\|(\mathfrak{R} * g)(T)\| + b_T^{-1}m(T)^{-1}[C_g\iota^{-1}T^\iota + \|g(T)\|].$$

3.3. Solvability and stability

In this section, we assume that:

(F1) The function $g : L^2(\Sigma) \to L^2(\Sigma)$ satisfies $g(0) = 0$ is locally Lipschitzian, that is

$$\|g(\nu_1) - g(\nu_2)\| \le \mathfrak{L}_g(\vartheta)\|\nu_1 - \nu_2\|, \ \forall \|\nu_1\|, \|\nu_2\| \le \vartheta,$$

where $\vartheta \ge 0$ and $\mathfrak{L}_g(\cdot)$ is a nonnegative function obeying that

$$\mathfrak{L}_g^* := \limsup_{\vartheta \to 0} \mathfrak{L}_g(\vartheta) < +\infty.$$

(F2) The real-valued function b defined on $[0,T]$ is continuous on $[0,T]$, nonnegative and satisfies that $b_T := \int_0^T b(\zeta)d\zeta > 0$.

(F3) The function $\xi : C([0,T];L^2(\Sigma)) \to \mathbb{H}^1$ satisfies $\xi(0) = 0$ and the locally Lipschitz condition

$$\|\xi(\nu_1) - \xi(\nu_2)\|_{\mathbb{H}^1} \le \mathfrak{L}_\xi(\vartheta)\|\nu_1 - \nu_2\|_\infty, \ \forall \|\nu_1\|_\infty, \|\nu_2\|_\infty \le \vartheta,$$

here $\mathfrak{L}_\xi(\cdot)$ is a nonnegative function such that $\mathfrak{L}_\xi^* := \limsup_{\vartheta \to 0} \mathfrak{L}_\xi(\vartheta) < +\infty$.

We define the mild solution of this problem as follows.

Definition 3.1. The pair $(a, u) \in \mathbb{H}^{-1} \times C([0,T];L^2(\Sigma))$ is said to be a mild solution of the problem (3.1)-(3.4) iff

$$a = \mathcal{S}\big[\xi(u) - \mathfrak{R}(T)\phi - \int_0^T \mathfrak{R}(T-\zeta)g(u(\zeta))d\zeta\big],$$
$$u(t) = \mathfrak{R}(t)\phi + \int_0^t \mathfrak{R}(t-\zeta)ab(\zeta)d\zeta + \int_0^t \mathfrak{R}(t-\zeta)g(u(\zeta))d\zeta.$$

The following theorem states a result on the existence of mild solution.

Theorem 3.1. *Under the assumptions (F1)-(F3), there exists $\delta > 0$ such that for $\|\phi\| < \delta$, the problem* (3.1)-(3.4) *has a unique mild solution, provided that*

$$\mathfrak{L}_b \mathfrak{L}_\xi^* + (\mathfrak{L}_b \lambda_1^{\frac{1}{2}} + 1)\lambda_1^{-1}\mathfrak{L}_g^* < 1,$$

where

$$\mathfrak{L}_b = \lambda_1^{-\frac{1}{2}} \|b\|_\infty b_T^{-1}(\lambda_1^{-1} + m(T)^{-1}). \tag{3.8}$$

We now consider the case ξ does not depend on u. We will prove that the solution map $(\xi, \phi) \mapsto (a, u)$ is Lipschitzian as a correspondence from $\mathbb{H}^1 \times L^2(\Sigma)$ to $\mathbb{H}^{-1} \times C([0,T]; L^2(\Sigma))$.

Theorem 3.2. *Assume that ξ is independent of u and g is globally Lipschitzian with Lipschitz constant $\mathfrak{L}_g^*$. Then the solution map $(\xi, \phi) \mapsto (a, u)$ is Lipschitz continuous, provided that:*

$$(\mathfrak{L}_b \lambda_1^{\frac{1}{2}} + 1)\lambda_1^{-1}\mathfrak{L}_g^* < 1.$$

3.4. Regularity analysis

If ξ is regular enough, we can show that the mild solution is a strong one.

Definition 3.2. A pair $(a, u) \in L^2(\Sigma) \times C([0,T]; L^2(\Sigma))$ is called a strong solution to the problem (3.1)-(3.4) iff (3.1), (3.3) and (3.4) hold as equations in $L^2(\Sigma)$.

To deal with strong solution, we replace (M) by a stronger hypothesis:

(M*) *The condition* (M) *holds and k is a nonincreasing function.*

The main result of this section is stated in the following theorem.

Theorem 3.3. *Assume that* (F1)-(F3) *and* (M*) *hold. If ξ takes values in $\mathbb{H}^2$ and b is Hölder continuous, then the mild solution of* (3.1)-(3.4) *is a strong one.*

In order to prove this result, we need some lemmas.

Lemma 3.3. *Assume that* (F1)-(F3) *and* (M*) *hold. Let (a, u) be the mild solution of the problem* (3.1)-(3.4). *Then u is Hölder continuous on $(0, T]$. Precisely, there exist $d_1, d_2 > 0$ and $\iota \in (0, \frac{1}{2})$ such that for all $t > 0$, $\omega \in (0, T - t]$ it holds that*

$$\|u(t + \omega) - u(t)\| \le (d_1 t^{-\iota} + d_2)\omega^\iota. \tag{3.9}$$

Lemma 3.4. *Let the assumptions of Theorem 3.3 hold and (a, u) be the mild solution of* $(3.1) - (3.4)$. *Then u is differentiable on $(0, T]$, that is $u'(t) \in L^2(\Sigma)$ $\forall t \in (0, T]$.*

Lemma 3.5. *Let the assumptions of Theorem 3.3 hold and (a, u) be the mild solution of* $(3.1) - (3.4)$. *Then* $\Delta u(t) \in L^2(\Sigma)$ $\forall t \in (0, T]$.

Lemma 3.6. *Let the assumptions of Theorem 3.3 hold and (a, u) be the mild solution of* $(3.1) - (3.4)$. *Then* $\mathfrak{D}_{t,\{k\}} \Delta u(t) \in L^2(\Sigma)$ $\forall t \in (0, T]$.

3.5. Examples of g and ξ

In this section, we deliver examples for (F1) and (F3).

For (F1), we choose $g : L^2(\Sigma) \to L^2(\Sigma)$ as follows:

$$g(v)(x) = \left(\int_\Sigma |v(x)|^2 dx\right)^p v(x) = \|v\|_{L^2(\Sigma)}^{2p} v(x), p \ge 1.$$

For (F3), we choose $\xi(u)(x) = \int_\Sigma \kappa(x,y)u(T,y)dy$, here $\kappa : \Sigma \times \Sigma \to \mathbb{R}$ is continuously differentiable with respect to the first variable, $\kappa(x,y) = 0$ when $x \in \partial\Sigma$ and we need the following condition

$$\int_\Sigma \|\nabla \kappa(x,y)\|^2_{L^2(\Sigma)} dy < +\infty.$$

Chapter 4

FINAL VALUE PROBLEM FOR RAYLEIGH-STOKES TYPE EQUATIONS INVOLVING WEAK-VALUED NONLINEARITIES

The final value problem governed by Rayleigh-Stokes type equations is investigated in the circumstance that the nonlinearity function may take values in Hilbert scales of negative order. We prove the existence and Hölder regularity results by analyzing the regularity of the resolvent operators and using the fixed point arguments. The content of this chapter is based on the paper [3] in the author's works related to the thesis that have been published.

4.1. Problem setting

Let $\Sigma \subset \mathbb{R}^d$ be a bounded domain with smooth boundary $\partial\Sigma$. Consider the following problem:

$$\frac{\partial u}{\partial t} - \Delta u - \mathfrak{D}_{t,\{k\}}\Delta u = g(u) \text{ in } \Sigma, 0 < t < T, \tag{4.1}$$

$$u = 0 \text{ on } \partial\Sigma,\ t > 0, \tag{4.2}$$

$$u(T, \cdot) = \xi \text{ in } \Sigma, \tag{4.3}$$

where g is given function, u is the unknown function defined on $(0, T] \times \Sigma$, $\mathfrak{D}_{t,\{k\}}$ is the nonlocal differential operator of Riemann-Liouville type defined by:

$$\mathfrak{D}_{t,\{k\}}v(t) = \frac{d}{dt}\int_0^t k(t-\zeta)v(\zeta)d\zeta,$$

with respect to the kernel function $k \in L^1_{loc}(\mathbb{R}^+)$. In this problem, we need the following conditions:

(M) *$k \in L^1_{loc}(\mathbb{R}^+)$ is nonnegative and $\mathfrak{a}(t) = 1 + k(t)$ is completely positive.*

(NND) *There exists a positive nonincreasing function $m \in L^1_{loc}(\mathbb{R}^+)$ such that*

$$m * \mathfrak{a} = 1 \text{ on } \mathbb{R}^+.$$

4.2. Representation of the mild solution and estimations for resolvent operator

We consider the resolvent operator $\mathfrak{R}(t)$ as in (1.7). The following lemma give us the properties of $\mathfrak{R}(t)$:

Lemma 4.1. *Let $\{\mathfrak{R}(t)\}_{t\geq 0}$ be a family of resolvent operators defined by (1.7), $u \in L^2(\Sigma)$ and $T > 0$. Then*

(a) $\mathfrak{R}(\cdot)u \in C([0,T]; L^2(\Sigma))$ and $\|\mathfrak{R}(t)\| \leq r(t,\lambda_1)$ for all $t \geq 0$.

(b) If k is nonincreasing, then $\mathfrak{R}(\cdot)u \in C^1((0,T]; L^2(\Sigma))$ and the following holds:

$$\|\mathfrak{R}'(t)\| \leq t^{-1} \text{ for all } t > 0. \tag{4.4}$$

(c) For $\nu \in (0,1)$, $\varpi > 0$ and $h \in C([0,T]; \mathbb{H}^{\varpi-1-\nu})$, we have:

$$\|\mathfrak{R} * h(t)\|_{\varpi}^2 \leq \int_0^t (t-\zeta)^{-\nu} \|h(\zeta)\|_{\varpi-1-\nu}^2 d\zeta,$$

and

$$\|\mathfrak{R} * h(t)\|_{\varpi}^2 \leq \int_0^t [1 * k(t-\zeta)]^{-\nu} \|h(\zeta)\|_{\varpi-1-\nu}^2 d\zeta.$$

(d) If $(1*k)^{-1} \in L^1(0,T)$ then for $\varpi > 0$ and $h \in C([0,T]; \mathbb{H}^{\varpi-2})$ we get

$$\|\mathfrak{R} * h(t)\|_{\varpi}^2 \leq \int_0^t \frac{\|h(\zeta)\|_{\varpi-2}^2}{(1*k)(t-\zeta)} d\zeta.$$

We now find a representation of the solution to the linear final value problem:

$$\frac{\partial u}{\partial t} - \Delta u - \mathfrak{D}_{t,\{k\}} \Delta u = \mathbb{F} \text{ in } \Sigma, 0 < t \leq T, \tag{4.5}$$

$$u = 0 \text{ on } \partial\Sigma,\ 0 < t \leq T, \tag{4.6}$$

$$u(T,\cdot) = \xi \text{ in } \Sigma, \tag{4.7}$$

where $\mathbb{F} \in C([0,T]; L^2(\Sigma))$. From (1.6) we have

$$\xi = \mathfrak{R}(T)\phi + \int_0^T \mathfrak{R}(T-\zeta)\mathbb{F}(\zeta) d\zeta.$$

Then

$$\phi = \mathfrak{R}(T)^{-1}[\xi - \int_0^T \mathfrak{R}(T-\zeta)\mathbb{F}(\zeta) d\zeta].$$

So the solution to (4.5)-(4.7) is given by

$$u(t) = \mathbb{S}(t)[\xi - \mathfrak{R} * \mathbb{F}(T)] + \mathfrak{R} * \mathbb{F}(t), \tag{4.8}$$

where

$$\mathbb{S}(t) = \mathfrak{R}(t)\mathfrak{R}(T)^{-1} = \sum_{i=1}^{+\infty} \frac{r(t,\lambda_i)}{r(T,\lambda_i)} \langle \cdot, e_i \rangle e_i. \tag{4.9}$$

Some important properties of $\mathbb{S}(t)$ are presented in the following proposition.

Lemma 4.2. *The family of operators $\mathbb{S}(t)$ defined by (4.9) has the following properties:*

(a) If $0 \leq \eta \leq 1$, $\varpi > 0$ and $\xi \in \mathbb{H}^{\varpi+2(1-\eta)}$ then

$$\|\mathbb{S}(t)\xi\|_{\varpi} \leq C(m) t^{-\eta} \|\xi\|_{\varpi+2(1-\eta)}, t > 0,$$

and

$$\|\mathbb{S}(t)\xi\|_{\varpi} \leq \frac{C(m)\|\xi\|_{\varpi+2(1-\eta)}}{[1*k(t)]^{\eta}}, t > 0,$$

where $C(m) = \lambda_1^{-1} + m(T)^{-1}$.

(b) *For* $0 < \nu < 1$, $0 \le \eta \le 1$, $\varpi > 0$ *and* $h \in C([0,T];\mathbb{H}^{\varpi+1-2\eta-\nu})$, *we have:*

$$\|\mathbb{S}(t)[\mathfrak{R} * h(T)]\|_{\varpi}^2 \le C(m)^2 t^{-2\eta} \int_0^T (T-\zeta)^{-\nu} \|h(\zeta)\|_{\varpi+1-2\eta-\nu}^2 d\zeta.$$

(c) *If* $0 \le \eta \le 1$ *and* $(1*k)^{-1} \in L^1(0,T)$ *then for* $\varpi > 0$ *and* $h \in C([0,T];\mathbb{H}^{\varpi-2\eta})$, *we have:*

$$\|\mathbb{S}(t)[\mathfrak{R} * h(T)]\|_{\varpi}^2 \le \frac{C(m)^2}{1*k(t)^{2\eta}} \int_0^T \frac{\|h(\zeta)\|_{\varpi-2\eta}^2}{1*k(T-\zeta)} d\zeta.$$

Lemma 4.3. *Assume that* k *is nonincreasing. Then for* $0 < \tau \le 1$, $0 < \omega \le 1$, $\varpi > 0$ *and* $\xi \in \mathbb{H}^{\varpi+2\omega}$, *there exists* $C_\tau > 0$ *such that*

$$\|[\mathbb{S}(t+\varrho) - \mathbb{S}(t)]\xi\|_{\varpi} \le \frac{C_\tau \varrho^{\tau\omega} \|\xi\|_{\varpi+2\omega}}{[1*k(t)]^{1-\omega} t^{\tau\omega}},\ t \in (0, T-\varrho], \varrho > 0.$$

4.3. Solvability

To deal with the problem (4.1)-(4.3), we use the following hypothesis for g:

(G) *The function* $g : \mathbb{H}^{\varpi} \to \mathbb{H}^{-\gamma}$ *satisfies* $g(0) = 0$, *here* $\varpi > 0$ *and* γ *is nonnegative. Moreover, there exist nonnegative functions* $\mathfrak{L}_g$ *and* $\mathcal{L}_g$ *such that for* $u_1, u_2 \in \mathbb{H}^{\varpi}$ *we have:*

$$\|g(u_1) - g(u_2)\|_{-\gamma} \le \mathfrak{L}_g(\|u_1\|_{\varpi}, \|u_2\|_{\varpi})\|u_1 - u_2\|_{\varpi}$$

and

$$\mathfrak{L}_g(\lambda\theta, \lambda\theta') \ge \mathcal{L}_g(\lambda)\mathfrak{L}_g(\theta, \theta'),\ \textit{for all}\ \lambda, \theta, \theta' > 0.$$

Based on (4.8), we give the concept of mild solution to the problem (4.1)-(4.3) as follows.

Definition 4.1. Given $\varpi > 0$. A function $u \in C((0,T];\mathbb{H}^{\varpi})$ is called a mild solution to the problem (4.1)-(4.3) if

$$u(t) = \mathbb{S}(t)\xi - \mathbb{S}(t)\int_0^T \mathfrak{R}(T-\zeta)g(u(\zeta))d\zeta + \int_0^t \mathfrak{R}(t-\zeta)g(u(\zeta))d\zeta,\ \forall t \in (0,T].$$

We consider the solution space of (4.1)-(4.3) as follows:

$$\mathcal{M}^{\varpi,\eta} = \{u \in C((0,T];\mathbb{H}^{\varpi}) : u(T) = \xi \text{ and } \|u\|_{\varpi,\eta} := \sup_{t>0} t^{\eta}\|u(t)\|_{\varpi} < +\infty\},$$

where $\varpi > 0, \eta > 0$ and ξ is given.

Theorem 4.1. *Given* $\varpi \in (0,1]$, $\eta \in (0,1)$, $\nu \in (0,1)$ *such that* $2\eta + \nu \ge \varpi + 1$. *Assume that* (G) *holds for* $\gamma = 2\eta + \nu - \varpi - 1$. *Then there exist* $\theta^* > 0$ *and* $\delta > 0$ *such that if* $\|\xi\|_{\varpi+2(1-\eta)} \le \delta$ *and*

$$6\mathfrak{L}_g^{*2}[C(m)^2\Lambda(T) + \lambda_1^{2\eta-2} \sup_{t\in(0,T]} t^{2\eta}\Lambda(t)] < 1,$$

here

$$\mathfrak{L}_g^* = \limsup_{\theta,\theta'\to 0} \mathfrak{L}_g(\theta,\theta'),$$

$$\Lambda(t) = \int_0^t (t-\zeta)^{-\nu}\zeta^{-2\eta}\mathcal{L}_g(\zeta^{\eta})^{-2}d\zeta,$$

then the problem (4.1)-(4.3) *has a unique mild solution* u *in* $\mathcal{M}^{\varpi,\eta}$ *satisfying that* $\|u\|_{\varpi,\eta} \le \theta^*$.

If g is globally Lipschitzian and the kernel k satisfies that $(1*k)^{-1} \in L^1(0,T)$ then we obtain the following result:

Theorem 4.2. *Assume that $g : \mathbb{H}^{\varpi} \to \mathbb{H}^{\varpi-2\eta}$, $\varpi > 0$, $0 < \eta < 1$ and g satisfies that*

$$\|g(u_1) - g(u_2)\|_{\varpi-2\eta} \leq \mathfrak{L}_g^* \|u_1 - u_2\|_\varpi \text{ for all } u_1, u_2 \in \mathbb{H}^{\varpi}.$$

If $(1 * k)^{-1} \in L^1(0,T)$ and

$$\mathfrak{L}^* := 2\mathfrak{L}_g^{*2}[C(m)^2 \Theta_\eta(T) + \lambda_1^{4(\eta-1)} \sup_{0<t\leq T} 1 * k(t)^{2\eta} \Theta_\eta(t)] < 1,$$

here

$$\Theta_\eta(t) = \int_0^t [1 * k(t-\zeta)]^{-1} [1 * k(\zeta)]^{-2\eta} d\zeta, \tag{4.10}$$

then the problem (4.1)-(4.3) *has a unique mild solution in the space*

$$\mathcal{N}^{\varpi,\eta} = \{u \in C((0,T]; \mathbb{H}^{\varpi}) : u(T) = \xi \text{ and } \|u\|_{\varpi,k,\eta} := \sup_{0<t\leq T} 1 * k(t)^\eta \|u(t)\|_\varpi < +\infty\},$$

for given $\xi \in \mathbb{H}^{\varpi+2(1-\eta)}$.

4.3. The Hölder regularity

In this section, we need a stronger hypothesis.

(M*) *The function k satisfies the hypothesis* (M) *and k is nonincreasing.*

Theorem 4.3. *Assume that* (M*) *holds and $g : \mathbb{H}^{\varpi} \to \mathbb{H}^{\varpi-2(1-\omega)}$, $\varpi > 0$, $0 < \omega < 1$ such that*

$$\|g(u_1) - g(u_2)\|_{\varpi-2(1-\omega)} \leq \mathfrak{L}_g^* \|u_1 - u_2\|_\varpi, \text{ for all } u_1, u_2 \in \mathbb{H}^{\varpi},$$

here $\mathfrak{L}_g^* > 0$. Moreover, we assume that $(1 * k)^{-1} \in L^1(0,T)$ and

$$\mathfrak{L}^* := 6\mathfrak{L}_g^{*2}[C(m)^2 \Theta_{1-\omega}(T) + \lambda_1^{-4\omega} \sup_{0<t\leq T} [1 * k(t)]^{2(1-\omega)} \Theta_{1-\omega}(t)] < 1,$$

$$6\lambda_1^{-4\omega} \mathfrak{L}_g^{*2} \mathcal{M}_{\omega,\tau}^* < 1, \;\; \Theta^* := \sup_{\varrho \in (0,T)} \frac{1}{\varrho^{2\tau\omega}} \left(\Theta_{1-\omega}(\varrho) + \int_0^\varrho \frac{d\zeta}{1 * k(\zeta)} \right) < +\infty,$$

with $\Theta_{1-\omega}$ defined by (4.10)*, $\tau \in (0,1]$ and*

$$\mathcal{M}_{\omega,\tau}^* := \sup_{t \in (0,T)} [1 * k(t)]^{2(1-\omega)} t^{2\tau\omega} \int_0^t \frac{d\zeta}{[1 * k(t-\zeta)][1 * k(\zeta)]^{2(1-\omega)} \zeta^{2\tau\omega}}.$$

Then the problem (4.1)-(4.3) *has a unique solution in $\mathcal{N}^{\varpi,1-\omega}$ for given $\xi \in \mathbb{H}^{\varpi+2\omega}$ and this solution is Hölder continuous on $(0,T]$.*

4.4. Examples

In this section, we give examples for g so that we have the conditions in Theorem 4.1, Theorem 4.2 and Theorem 4.3.

Example 4.1. Example for Theorem 4.1.
Assume that $\Sigma \subset \mathbb{R}^d$ with $d \geq 2$. Then we can choose $g(u) = |u|^r$ and $\varpi, \gamma, \nu, \eta$ as in Theorem 4.1.

Example 4.2. Example for Theorem 4.2 and Theorem 4.3.
Consider $\Sigma \subset \mathbb{R}^d$ with $d \geq 3$. Then we can choose g such that it is globally Lipschitzian:

$$g(u) = \mathbf{g}(x) \cdot \nabla u + b(x) \ln(1 + u^2),$$

where $\mathbf{g} = (\mathbf{g}_1, ..., \mathbf{g}_d) \in (L^\infty(\Sigma))^d$, $b \in L^d(\Sigma)$, and

$$k(t) = \frac{t^{-\alpha}}{\Gamma(1-\alpha)}$$

with $0 < \alpha < 1, \eta = \omega = \frac{1}{2}$.

CONCLUSION

1. Results of the thesis

We study forward and inverse problems for Rayleigh-Stokes equation, which is a nonlocal partial differential equation. Our results are:

(a) For the Cauchy problem for Rayleigh-Stokes equation involving delays:

- Solvability: the existence of mild solution.
- Stability: the existence of a bounded absorbing set, the asymptotic stability of zero solution and the existence of a compact set of decay solutions.

(b) For the identification problem for generalized Rayleigh-Stokes equation:

- The existence and uniqueness of mild solution and the Lipschitz continuity of the solution map.
- The mild solution becomes strong solution with suitable conditions.

(c) For the final value problem for Rayleigh-Stokes type equations involving weak-valued nonlinearities:

- The existence and uniqueness of mild solution.
- The Hölder regularity of mild solution.

2. Recommendation

- The Cauchy problem for generalized Rayleigh-Stokes equation: find the conditions such that the mild solution continuously depends on the kernel k.
- The identification problem for generalized Rayleigh-Stokes equation: consider other cases of measurement, for example $\frac{1}{T}\int_0^T u(s)ds = \xi$.
- The final value problem for generalized Rayleigh-Stokes equation: find the conditions such that the mild solution becomes strong solution.

AUTHOR'S WORKS RELATED TO THE THESIS THAT HAVE BEEN PUBLISHED

[1] D. Lan, P.T. Tuan, (2022), On stability for semilinear generalized Rayleigh-Stokes equation involving delays, *Quarterly of Applied Mathematics* 80, 701–715. (SCI-E/Q2)

[2] T.D. Ke, L.T.P. Thuy, P.T. Tuan, (2022), An inverse source problem for generalized Rayleigh-Stokes equations involving superlinear perturbations, *Journal of Mathematical Analysis and Applications* 507:2 , 125797. (SCI-E/Q1)

[3] P.T. Tuan, T.D. Ke, N.N. Thang, (2023), Final value problem for Rayleigh-Stokes type equations involving weak-valued nonlinearities, *Fractional Calculus and Applied Analysis* 26, 694–717. (SCI-E/Q1)