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THE BOUNDEDNESS OF HAUSDORFF TYPE OPERATORS ON SOME FUNCTION SPACES

SUMMARY OF DOCTORAL THESIS IN MATHEMATICS

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INTRODUCTION

1. Motivation and history of the problem

The study of the boundedness of operators T on some functional spaces is one of the important topics in harmonic analysis. More specifically, we need to prove the following inequality

$$\|Tf\|_Y \leq C\|f\|_X, \tag{1}$$

where C is a positive constant, and X, Y are two functional with norms $\|\cdot\|_X$ and $\|\cdot\|_Y$, respectively. To illustrate the importance of this problem, let us recall several known problem as follows.

• Lebesgue's differentiation theorem reads that for any locally integrable function f on $\mathbb{R}^n$, we have

$$\lim_{r\to 0} \frac{1}{|B(x,r)|} \int_{B(x,r)} f(y)dy = f(x)$$

for almost everywhere x in $\mathbb{R}^n$. To prove the Lebesgue's differentiation theorem, we consider the centered Hardy-Littlewood maximal function as follow

$$Mf(x) = \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)|dy,$$

and prove that the centered Hardy–Littlewood maximal function is of weak type $(1,1)$. Moreover, it is defined the Hardy–Littlewood maximal function as follows

$$\mathcal{M}f(x) = \sup_{x\in B} \frac{1}{|B|} \int_B |f(y)|dy,$$

where the supremum is taken over all balls B in $\mathbb{R}^n$.

• Next, let us consider the following Dirichlet problem

$$\begin{cases} \sum_{i=1}^{n} \partial_{x_i}^2 u(x,t) + \partial_t^2 u(x,t) = 0, \quad \text{for} \quad (x,t) \in \mathbb{R}^n \times \mathbb{R}^+, \\ u(x,0) = f(x), \quad \text{for almost all} \quad x \in \mathbb{R}^n, \end{cases}$$

where f in $L^p(\mathbb{R}^n)$ for $1 \le p < \infty$. To solve the Dirichlet problem, we find the solution in terms of the form

$$u(x,t) = (f * P_t)(x),$$

where $P_t(x) = t^{-n}P(t^{-1}x)$, and

$$P(x) = \frac{\Gamma\left(\frac{n+1}{2}\right)}{\pi^{\frac{n+1}{2}}} \frac{1}{(1+|x|^2)^{\frac{n+1}{2}}}$$

is the Poisson kernel. Clearly, $P_t(x_1, ..., x_n, t)$ is the harmonic function of the variables $(x_1, ..., x_n, t)$, that is,

$$\sum_{i=1}^{n} \partial_i^2 P_t + \frac{d^2}{dt^2} P_t = 0.$$

Therefore, the function $u(x,t)$ is also harmonic in $\mathbb{R}^n \times \mathbb{R}^+$, and converges to $f(x)$ in $L^p(\mathbb{R}^n)$ as t tends to 0. To complete the proof, it remains to show that $u(x,t)$ converges to $f(x)$ a.e. as t tends to 0. However, it is easily showed from the inequality

$$\sup_{t>0} |u(x,t)| \le \mathscr{M} f(x),$$

and the weak type (p,p) property of the Hardy–Littlewood maximal function.

• Finally, we consider a Cauchy problem to the Schrödinger equation as follows

$$\begin{cases} i\partial_t u(x,t) - \Delta u(x,t) = 0, & (x,t) \in \mathbb{R}^n \times \mathbb{R}^+, \\ u(x,0) = u_0(x). \end{cases}$$

As we know, the solution of this problem is given by $u(x,t) = (e^{-it\Delta}u_0)(x)$. Note that $u(x,t) = (e^{-it\Delta}u_0)(x)$ is defined through its Fourier transform

$$\widehat{(e^{-it\Delta}u_0)}(\xi) = e^{it|\xi|^2}\hat{u_0}(\xi).$$

To study regularity of the solution, we need to estimate

$$||e^{-it\Delta}(u_0 - v_0)||_Y \le C||u_0 - v_0||_X,$$

But, since the operator $e^{-it\Delta}$ is linear, we only prove that

$$\|e^{-it\Delta}f\|_Y \leq C\|f\|_X.$$

From all above cases, we have partly shown the importance of the study on the boundedness of operators on some functional spaces to solve problems in analysis as well as partial differential equations.

Moreover, another important problem is to give some necessary and sufficient conditions for. Also, we may show the best constant C in the inequality (1). For some important operators in harmonic analysis, such as the Hardy–Littlewood maximal functions, this is a very difficult problem. More details, L. Grafakos and S. M. Smith (1997), D. Melas (2003) can be found and their references.

In 1920, G. H. Hardy established the following integral inequality

$$\|\mathscr{H}f\|_{L^p(\mathbb{R}^+)} \leq \frac{p}{p-1}\|f\|_{L^p(\mathbb{R}^+)},$$

where $1 < p < \infty$ and f is a non-negative measurable function on $(0;\infty)$. Furthermore, the constant $\frac{p}{p-1}$ is sharp. Here, $\mathscr{H}$ is the Hardy operator defined by

$$\mathscr{H}f(x) = \frac{1}{x}\int_0^x f(t)dt.$$

Hardy's inequality and its extended forms play an important role in the theory of partial differential equations, the theory of approximation, the theory of functional spaces (see K.F. Andersen and B. Muckenhoupt (1982), D.E. Edmunds and W.D. Evans (2004), D. Lukkassena, A. Meidella, L.E. Persson and N. Samko (2012)).

One of the important operators in harmonic analysis is the Hausdorff operator, which is closely related to the problem of the sums of the classical Fourier series. Let Φ be a locally integrable function on $(0,\infty)$. The one-dimensional Hausdorff operator is defined as follows

$$\mathscr{H}_\Phi f(x) = \int_0^\infty \frac{\Phi(t)}{t} f\left(\frac{x}{t}\right) dt. \tag{2}$$

Clearly, when $\Phi(t) = \frac{\chi_{(1,\infty)}(t)}{t}$, the Hausdorff operator reduces to the Hardy operator above. Furthermore, it is worth pointing out that if the kernel function U is taken appropriately, then the Hausdorff operator also reduces to many other classical operators in analysis such as the Cesàro operator, Hardy—Littlewood—Pólya operator, Riemann—Liouville fractional integral operator and Hardy–Littlewood average operator (see K. Andersen and E. Sawyer (1988), J. Chen, D. Fan and J. Li (2012), M. Christ and L. Grafakos. (1995), Z. W Fu, S. L Gong, S. Z Lu and W. Yuan (2015), A. Miyachi (2004)).

The Hausdorff operator is extended to the space $\mathbb{R}^n$ by Brown and Móricz (2002), and is independent by Lerner and Liflyand (2007). To be more precise, let φ be a locally integrable function on $\mathbb{R}^n$. The Hausdorff $\mathscr{H}_{\varphi,A}$ operator associated to the kernel function φ defined by

$$\mathscr{H}_{\varphi,A}f(x) = \int_{\mathbb{R}^n} \varphi(t)f(A(t)x)dt, \quad x \in \mathbb{R}^n,$$

where $A(t)$ is an $n \times n$ matrix satisfying $\det A(t) \neq 0$ for almost everywhere t in the support of φ. If we take the matrix $A(t)$ and the function φ appropriately, then $\mathscr{H}_{\varphi,A}$ reduces to some known operators in analysis. For more details see E. Liflyand (2013) and the references therein.

In recent years, the Hausdorff operators and their commutators, both linear and multilinear, have been significantly studied by many mathematicians over the world. Some necessary and sufficient conditions for the boundedness of the Hausdorff operators and their commutators have been established. At the same time, the norm of the operators are estimated. More details, we can be found in K. F Andersen (2003), R. Bandaliyev and P. Gorka (2019), G. Brown and F. Móricz (2002), V.I Burenkov and E. Liflyand (2020), N.M Chuong, D.V Duong and K.H Dung (2018, 2019), J. Chen, J. Dai, D. Fan and X. Zhu (2018), J. Chen, D. Fan and J. Li (2012), J.H Guo (2015), A. Hussain and G. Gao (2013), A. Hussain and M. Ahmed (2017), Y. Kanjin (2001), J. C Kuang (2012), A. Lerneran and E. Liflyand (2007), S.S Volosivets (2013, 2017), Q. Wu, D. Fan (2017).

2. Purpose of thesis

This thesis investigates some necessary and sufficient conditions for the boundedness of some Hausdorff type operators and their commutators on the real field and on the Heisenberg group. Moreover, the estimates for operator norms in each case are worked out.

3. Object and scope of thesis

The airm of this thesis is to study the boundedness of the Hausdorff type operators and their commutators on the real field and the Heisenberg group. The scope of the thesis is included the following contents:

• **Content 1:** We study the necessary and sufficient conditions for the boundness of the rough Hausdorff operator $\mathscr{H}_{\Phi,\Omega}$ on the central Morrey spaces, the Herz space, and the Morrey-Herz space with homogeneous weight. Also, we establish the boundedness of the commutators of rough Hausdorff operators on the two weighted Morrey-Herz type spaces with their symbols belonging to Lipschitz space.

• **Content 2:** We study the necessary and sufficient conditions for the boundness of the multilinear Hausdorff operator $\mathscr{H}_{\Phi,\vec{A}}$ on the product of some two weighted function spaces such as the two weighted Morrey, Herz and Morrey-Herz spaces both power weight and the Muckenhoupt weights.

• **Content 3:** We give the sufficient conditions for the boundness of the commutator of rough Hausdorff operators $\mathscr{H}^b_{\Phi,\Omega}$ and matrix Hausdorff operator $\mathscr{H}^b_{\Phi,A}$ on the Heisenberg group with their symbols belonging to weighted ℓ-central BMO space on the central Morrey space, the Herz space, and the Morrey-Herz space with homogeneous weight as well as Muckenhoupt weight.

4. Research methods

• To study the boundedness of the Hausdorff type operators on real field and the Heisenberg group, we rely on the methods built by Coifman-Rochberg-Weiss (1976) on homogeneous spaces with characteristic transformations of powers and Muckenhoupt weights. We use some important inequalities in analysis, and uses the schema that Xiao (2001) developed, in which the test functions are chosen appropriately to obtain the estimates for the norm of the operator.

• For the study of commutator, it is based on the famous method of Coifman–Rochberg–Weiss (1976). The key is to estimate the average oscillation and use some techniques proposed by D. Fan, Chen, Li, Fu, Lu et al. (see (2011), (2012), (2018)).

5. Results of thesis

My thesis written from three papers in the published works related to the thesis.

• Study of the necessary and sufficient conditions for the boudedness of the rough Hausdorff operator $\mathscr{H}_{\Phi,\Omega}$ on the central Morrey space, the Herz space, and Morrey-Herz space with homogeneous weight. In addition, we have the norm estimate of the Hausdorff operator $\mathscr{H}_{\Phi,\Omega}$ on such spaces. Furthermore, we give sufficient conditions for the boundedness of commutators of the rough Hausdorff operator $\mathscr{H}^{b}_{\Phi,\Omega}$ with their symbols belonging to Lipschitz on the central Morrey space, Herz space, and Morrey-Herz space with two homogeneous weights. These are the main results of Chapter 2.

• Norm estimation of the multilinear Hausdorff operator $\mathscr{H}_{\Phi,\vec{A}}$ on the product of the central Morrey space, the Herz space, and the Morrey-Herz space with two power weights. As a conseqence, we have to estimate the norm of the multilinear Hardy-Ceàro operator on the product of such spaces. In addition, we give a sufficient condition for the boundedness of the multilinear Hausdorff operator $\mathscr{H}_{\Phi,\vec{A}}$ on the product of the central Morrey spaces, and the Morrey-Herz space with two Muckenhoupt weights. These are the main results of Chapter 3.

• Study of the sufficient conditions for the boudedness of commutators of the Hausdorff operator $\mathscr{H}^b_{\Phi,\Omega}$ and the matrix operator Hausdorff $\mathscr{H}^b_{\Phi,A}$ with the symbol in weighted ℓ-central BMO space on the central Morrey, Herz, and Morrey-Herz spaces with power or Muckenhoupt weights on the Heisenberg group. These are the main results of Chapter 4.

6. Structures of thesis

Beside Introduction, Conclusion, Authors works erences, the dissertation consists of four chapters as follows:

• Chapter 1 is introduced some concepts and background knowledge;

• Chapter 2 is devoted to the study of boundedness of the rough Hausdorff operator and its commutators on Morrey–Herz spaces;

• Chapter 3 is devoted to the study of two-weighted estimates for multilinear Hausdorff operators on the Morrey–Herz spaces;

• Chapter 4 is devoted to the study of some weighted estimates for commutators of Hausdorff type operators on the Heisenberg group.

Chapter 1

PRELIMINARIES

In this chapter, we present some concepts and results that will be used in the whole thesis.

1.1. Lebesgue space

In this section we present some concepts and background knpwledge about the Lebesgue space, Lebesgue convergence theorem, Hölder inequality, Minkowski's inequality, Fubini's Theorem.

1.2. Some symbols and function spaces

In this section we explain some symbols and recall definitions of weighted Morrey, Herz, Morrey-Herz type spaces.

1.3. Homogenous weights, power weights and Muckenhoupt weights

In this section, we recall the definition of homogeneous weights, power weights, Muckenhoupt weights as well as some the Lemmas, Propositions, which is used in the sequel.

1.4. Heisenberg group

In this section, we recall the Heisenberg group, as well as the definition of ℓ - central BMO space on $\mathbb{H}^n$.

Chapter 2

WEIGHTED MORREY–HERZ SPACE ESTIMATES FOR ROUGH HAUSDORFF OPERATOR AND ITS COMMUTATORS

In this chapter, we give necessary and sufficient conditions for the boundedness of rough Hausdorff operators $\mathcal{H}_{\Phi,\Omega}$ on Herz, Morrey and Morrey–Herz spaces with absolutely homogeneous weights. Especially, the estimates for operator norms in each case are worked out. As a consequence, we obtain some new estimates for the high dimensional Hardy operator and adjoint Hardy operator. Moreover, we also establish the boundedness of the commutators of rough Hausdorff operators $\mathcal{H}^b_{\Phi,\Omega}$ on the two weighted Morrey–Herz type spaces with their symbols belonging to Lipschitz space.

This chapter is written based on the paper [1].

2.1. Introduction

The problem of research is the norm estimate of the operator $\mathcal{H}_{\Phi,\Omega}$ and the boundedness of the $\mathcal{H}^b_{\Phi,\Omega}$ on Morrey-Herz type spaces.

2.2. The $\mathcal{H}_{\Phi,\Omega}$ operator and the power weights

In this section, we present the necessary and sufficient conditions for the boundedness of the operator $\mathcal{H}_{\Phi,\Omega}$ on some function spaces with homogeneous weights, such as weighted central Morrey space (Theorem 2.1), weighted Herz space (Theorem 2.2), weighted Morrey-Herz space (Theorem 2.3).

Theorem 2.1. *Let* $1 \le q < \infty, 1 + \lambda q > 0, \lambda \in \mathbb{R}, \gamma > -n$ *and* $\Omega \in L^{q'}(S_{n-1})$.
i) If $\omega(x') \ge c > 0$ *for all* $x' \in S_{n-1}$ *and*

$$\mathcal{C}_1 = \int_0^\infty \frac{|\Phi(t)|}{t^{1+(n+\gamma)\lambda}} dt < \infty,$$

we have $\mathcal{H}_{\Phi,\Omega}$ *is a bounded operator on* $\dot{M}^{\lambda,q}_{\omega}(\mathbb{R}^n)$*. Moreover,*

$$\|\mathcal{H}_{\Phi,\Omega}\|_{\dot{M}^{\lambda,q}_{\omega}(\mathbb{R}^n)\to\dot{M}^{\lambda,q}_{\omega}(\mathbb{R}^n)} \lesssim \|\Omega\|_{L^{q'}(S_{n-1})}.\mathcal{C}_1.$$

ii) Conversely, suppose $\Omega \in L^{q'}(S_{n-1}, \omega(x')d\sigma(x'))$*,* Φ *is a real function with a constant sign in* $\mathbb{R}^n$*. Then, if* $\mathcal{H}_{\Phi,\Omega}$ *is bounded on* $\dot{M}^{\lambda,q}_{\omega}(\mathbb{R}^n)$ *then* $\mathcal{C}_1 < \infty$*. Furthemore,*

$$\|\mathcal{H}_{\Phi,\Omega}\|_{\dot{M}^{\lambda,q}_{\omega}(\mathbb{R}^n)\to\dot{M}^{\lambda,q}_{\omega}(\mathbb{R}^n)} \geq \frac{\|\Omega\|^{q'}_{L^{q'}(S_{n-1})}}{\|\Omega\|^{\frac{q'}{q}}_{L^{q'}(S_{n-1},\omega(x')d\sigma(x'))}}.\mathcal{C}_1.$$

Theorem 2.2. *Let* $1 \leq p, q < \infty, \alpha \in \mathbb{R}, \gamma \in \mathbb{R}$ *and* $\Omega \in L^{q'}(S_{n-1})$*.*
i) If $\omega(x') \geq c > 0$ *for all* $x' \in S_{n-1}$ *and*

$$\mathcal{C}_2 = \int_0^{\infty} \frac{|\Phi(t)|}{t^{1-\frac{\gamma}{q}-\frac{n}{q}-\alpha}} dt < \infty,$$

we have $\mathcal{H}_{\Phi,\Omega}$ *is a bounded operator on* $\dot{K}^{\alpha,p,q}_{\omega}(\mathbb{R}^n)$*. Moreover,*

$$\|\mathcal{H}_{\Phi,\Omega}\|_{\dot{K}^{\alpha,p,q}_{\omega}(\mathbb{R}^n)\to\dot{K}^{\alpha,p,q}_{\omega}(\mathbb{R}^n)} \lesssim \|\Omega\|_{L^{q'}(S_{n-1})}.\mathcal{C}_2.$$

ii) Conversely, suppose $\Omega \in L^{q'}(S_{n-1}, \omega(x')d\sigma(x'))$*,* Φ *is a real function with a constant sign in* $\mathbb{R}^n$*. Then, if* $\mathcal{H}_{\Phi,\Omega}$ *is bounded on* $\dot{K}^{\alpha,p,q}_{\omega}(\mathbb{R}^n)$ *then* $\mathcal{C}_2 < \infty$*. Furthermore,*

$$\|\mathcal{H}_{\Phi,\Omega}\|_{\dot{K}^{\alpha,p,q}_{\omega}(\mathbb{R}^n)\to\dot{K}^{\alpha,p,q}_{\omega}(\mathbb{R}^n)} \geq \frac{\|\Omega\|^{q'}_{L^{q'}(S_{n-1})}}{\|\Omega\|^{\frac{q'}{q}}_{L^{q'}(S_{n-1},\omega(x')d\sigma(x'))}}.\mathcal{C}_2.$$

Theorem 2.3. *Let* $0 < p < \infty, 1 \leq q < \infty, \alpha \in \mathbb{R}, \gamma \in \mathbb{R}, \lambda > 0$ *and* $\Omega \in L^{q'}(S_{n-1})$*.*
i) If $\omega(x') \geq c > 0$ *for all* $x' \in S_{n-1}$ *and*

$$\mathcal{C}_3 = \int_0^{\infty} \frac{|\Phi(t)|}{t^{1-\frac{\gamma}{q}-\frac{n}{q}+\lambda-\alpha}} dt < \infty,$$

then $\mathcal{H}_{\Phi,\Omega}$ *is a bounded operator on* $M\dot{K}^{\alpha,\lambda,p,q}_{\omega}(\mathbb{R}^n)$*. Moreover,*

$$\|\mathcal{H}_{\Phi,\Omega}\|_{M\dot{K}^{\alpha,\lambda,p,q}_{\omega}(\mathbb{R}^n)} \lesssim \|\Omega\|_{L^{q'}(S_{n-1})}.\mathcal{C}_3.$$

ii) Conversely, suppose $\Omega \in L^{q'}(S_{n-1}, \omega(x')d\sigma(x'))$, Φ is a real function with a constant sign in $\mathbb{R}^n$. Then, if $\mathscr{H}_{\Phi,\Omega}$ is bounded on $M\dot{K}^{\alpha,\lambda,p,q}_{\omega}(\mathbb{R}^n)$ then $\mathscr{C}_3 < \infty$. Furthermore,

$$\|\mathscr{H}_{\Phi,\Omega}\|_{M\dot{K}^{\alpha,\lambda,p,q}_{\omega}(\mathbb{R}^n)\to M\dot{K}^{\alpha,\lambda,p,q}_{\omega}(\mathbb{R}^n)} \geq \frac{\|\Omega\|^{q'}_{L^{q'}(S_{n-1})}}{\|\Omega\|^{\frac{q'}{q}}_{L^{q'}(S_{n-1},\omega(x')d\sigma(x'))}}.\mathscr{C}_3.$$

By these, we obtain some results on the boundedness of the operator $\mathscr{H}_{\Phi,\Omega}$ on some spaces with powers weight such as weighted central Morrey spaces (Corollary 2.1), weighted Herz spaces (Corollary 2.2) and weighted Morrey-Herz spaces (Corollary 2.3).

Corollary 2.1. *Let $1 \leq q < \infty, 1+\lambda q > 0$ and $\lambda \in \mathbb{R}$. Suppose $\Omega \in L^{q'}(S_{n-1})$, $\omega(x) = |x|^{\gamma}$ with $\gamma > -n$ and Φ is a non-negative radius function. Then, $\mathscr{H}_{\Phi,\Omega}$ is a bounded on $\dot{M}^{\lambda,q}_{\omega}(\mathbb{R}^n)$ if and only if*

$$\mathscr{C}_{1.1} = \int_0^{\infty} \frac{\Phi(t)}{t^{1+(n+\gamma)\lambda}}dt < \infty.$$

Furthermore $\|\mathscr{H}_{\Phi,\Omega}\|_{\dot{M}^{\lambda,q}_{\omega}(\mathbb{R}^n)\to\dot{M}^{\lambda,q}_{\omega}(\mathbb{R}^n)} \simeq \|\Omega\|_{L^{q'}(S_{n-1})}.\mathscr{C}_{1.1}$.

Corollary 2.2. *Let $1 \leq p, q < \infty$, $\alpha \in \mathbb{R}$, $\Omega \in L^{q'}(S_{n-1})$ and $\omega(x) = |x|^{\gamma}$ with $\gamma \in \mathbb{R}$. Suppose Φ is a non-negative radius function. Then, $\mathscr{H}_{\Phi,\Omega}$ is a bounded on $\dot{K}^{\alpha,p,q}_{\omega}(\mathbb{R}^n)$ if and only if*

$$\mathscr{C}_{2.1} = \int_0^{\infty} \frac{\Phi(t)}{t^{1-\frac{\gamma}{q}-\frac{n}{q}-\alpha}}dt < \infty.$$

Furthermore $\|\mathscr{H}_{\Phi,\Omega}\|_{\dot{K}^{\alpha,p,q}_{\omega}(\mathbb{R}^n)\to\dot{K}^{\alpha,p,q}_{\omega}(\mathbb{R}^n)} \simeq \|\Omega\|_{L^{q'}(S_{n-1})}.\mathscr{C}_{2.1}$.

Remark that Corollary 2.2 is an extension of Theorem 3.1 of Chen, Fan and Li (2012) on Lebesgue space with power weight.

Corollary 2.3. *Let $0 < p < \infty, 1 \leq q < \infty, \alpha \in \mathbb{R}, \gamma \in \mathbb{R}$ and $\lambda > 0$. Suppose $\Omega \in L^{q'}(S_{n-1})$, $\omega(x) = |x|^{\gamma}$ and Φ is a non-negative radius function. Then, $\mathscr{H}_{\Phi,\Omega}$ is a bounded on $M\dot{K}^{\alpha,\lambda,p,q}_{\omega}(\mathbb{R}^n)$ if and only if*

$$\mathscr{C}_{3.1} = \int_0^{\infty} \frac{\Phi(t)}{t^{1-\frac{\gamma}{q}-\frac{n}{q}+\lambda-\alpha}}dt < \infty.$$

Furthermore $\|\mathscr{H}_{\Phi,\Omega}\|_{M\dot{K}^{\alpha,\lambda,p,q}_{\omega}(\mathbb{R}^n)\to M\dot{K}^{\alpha,\lambda,p,q}_{\omega}(\mathbb{R}^n)} \simeq \|\Omega\|_{L^{q'}(S_{n-1})}.\mathscr{C}_{3.1}$.

In particular, we obtain some new Hardy type inequalities for the high dimensional Hardy operator and the adjoint Hardy operator is an extension of the results of Christ and Grafakos (1995), on some spaces of power weight such as the weighted central Morrey spaces (Corollary 2.4), the weighted Herz spaces (Corollary 2.5), the weighted Morrey-Herz spaces (Corollary 2.6).

Corollary 2.4. *Let $1 \le q < \infty, 1+\lambda q > 0, \lambda \in \mathbb{R}$, and $\omega(x) = |x|^\gamma$ for $\gamma > -n$. Then, the high dimensional Hardy operator is a bounded operator on $\dot{M}^{\lambda,q}_\omega(\mathbb{R}^n)$ if and only if*

$$\mathscr{C}_{1.2} = \int_1^\infty \frac{1}{t^{n+1+(n+\gamma)\lambda}} dt < \infty.$$

Moreover, $\|\mathscr{H}\|_{\dot{M}^{\lambda,q}_\omega(\mathbb{R}^n)\to\dot{M}^{\lambda,q}_\omega(\mathbb{R}^n)} \simeq \mathscr{C}_{1.2}$. Similarly, the adjoint Hardy operator is a bounded operator on $\dot{M}^{\lambda,q}_\omega(\mathbb{R}^n)$ if and only if

$$\mathscr{C}_{1.3} = \int_0^1 \frac{1}{t^{1+(n+\gamma)\lambda}} dt < \infty.$$

Also, $\|\mathscr{H}^\star\|_{\dot{M}^{\lambda,q}_\omega(\mathbb{R}^n)\to\dot{M}^{\lambda,q}_\omega(\mathbb{R}^n)} \simeq \mathscr{C}_{1.3}$.

Corollary 2.5. *Let $1 \le p, q < \infty, \alpha \in \mathbb{R}$ and $\omega(x) = |x|^\gamma$ for $\gamma \in \mathbb{R}$. Then, the Hardy operator is a bounded operator on $\dot{K}^{\alpha,p,q}_\omega(\mathbb{R}^n)$ if and only if*

$$\mathscr{C}_{2.2} = \int_1^\infty \frac{1}{t^{n+1-\frac{\gamma}{q}-\frac{n}{q}-\alpha}} dt < \infty.$$

Moreover, $\|\mathscr{H}\|_{\dot{K}^{\alpha,p,q}_\omega(\mathbb{R}^n)\to\dot{K}^{\alpha,p,q}_\omega(\mathbb{R}^n)} \simeq \mathscr{C}_{2.2}$. Analogously, the adjoint Hardy operator is a bounded operator on $\dot{K}^{\alpha,p,q}_\omega(\mathbb{R}^n)$ if and only if

$$\mathscr{C}_{2.3} = \int_0^1 \frac{1}{t^{1-\frac{\gamma}{q}-\frac{n}{q}-\alpha}} dt < \infty.$$

Moreover, $\|\mathscr{H}^\star\|_{\dot{K}^{\alpha,p,q}_\omega(\mathbb{R}^n)\to\dot{K}^{\alpha,p,q}_\omega(\mathbb{R}^n)} \simeq \mathscr{C}_{2.3}$.

Corollary 2.6. *Let $0 < p < \infty, 1 \le q < \infty, \alpha \in \mathbb{R}, \gamma \in \mathbb{R}$, $\lambda > 0$ and $\omega(x) = |x|^\gamma$. Then, the Hardy operator is a bounded operator on $M\dot{K}^{\alpha,\lambda,p,q}_\omega(\mathbb{R}^n)$ if and only if*

$$\mathscr{C}_{3.2} = \int_1^\infty \frac{1}{t^{n+1-\frac{\gamma}{q}-\frac{n}{q}+\lambda-\alpha}} dt < \infty.$$

Also, $\|\mathscr{H}\|_{M\dot{K}^{\alpha,\lambda,p,q}_{\omega}(\mathbb{R}^n)\to M\dot{K}^{\alpha,\lambda,p,q}_{\omega}(\mathbb{R}^n)} \simeq \mathscr{C}_{3.2}$. *Similarly, the adjoint Hardy operator is bounded on* $M\dot{K}^{\alpha,\lambda,p,q}_{\omega}(\mathbb{R}^n)$ *if and only if*

$$\mathscr{C}_{3.3} = \int_0^1 \frac{1}{t^{1-\frac{\gamma}{q}-\frac{n}{q}+\lambda-\alpha}}dt < \infty.$$

Moreover, we get $\|\mathscr{H}^{\star}\|_{M\dot{K}^{\alpha,\lambda,p,q}_{\omega}(\mathbb{R}^n)\to M\dot{K}^{\alpha,\lambda,p,q}_{\omega}(\mathbb{R}^n)} \simeq \mathscr{C}_{3.3}$.

2.3. Commutator $\mathscr{H}^b_{\Phi,\Omega}$ and homogeneous weights

In this section, we give sufficient conditions for the boundedness of the commutator $\mathscr{H}^b_{\Phi,\Omega}$ with their symbols b belonging to Lipschitz space on some two weighted function spaces such as two weighted central Morrey spaces (Theorem 2.4), two weighted Herz spaces (Theorem 2.5), two weighted Morrey-Herz spaces (Theorem 2.6).

Theorem 2.4. *Let* $1 \leq q < \infty$, $\Omega \in L^{p'}(S_{n-1})$ *and* $b \in Lip^{\beta}(\mathbb{R}^n)$ *for* $0 < \beta \leq 1$. *Suppose* $v, \omega \in \mathscr{W}_{\gamma}$, $\gamma > -n$ *and* $\omega(x') \geq c > 0$ *for all* $x' \in S_{n-1}$. *If* $\lambda_1 = \lambda - \frac{\beta q}{n+\gamma} > 0$ *and*

$$\mathscr{C}_4 = \int_0^{\infty} \frac{|\Phi(t)|}{t^{1+(\gamma+n)\frac{\lambda_1-1}{q}}(1+t^{-1})^{-\beta}}dt < \infty,$$

then $\mathscr{H}^b_{\Phi,\Omega}$ *is a bounded operator from* $\dot{M}^{\lambda_1,q}_{v,\omega}(\mathbb{R}^n)$ *to* $\dot{M}^{\lambda,q}_{v,\omega}(\mathbb{R}^n)$.

Theorem 2.5. *Let* $1 \leq p, q < \infty$, $\Omega \in L^{q'}(S_{n-1})$ *and* $b \in Lip^{\beta}(\mathbb{R}^n)$ *for* $0 < \beta \leq 1$. *Suppose* $v, \omega \in \mathscr{W}_{\gamma}$, $\gamma > -n$ *and* $\omega(x') \geq c > 0$ *for all* $x' \in S_{n-1}$. *If* $\alpha_1 = \alpha_2 + \frac{n\beta}{n+\gamma}$ *and*

$$\mathscr{C}_5 = \int_0^{\infty} \frac{|\Phi(t)|}{t^{1-\frac{\gamma}{q}-\frac{n}{q}-\alpha_1\left(1+\frac{\gamma}{n}\right)}(1+t^{-1})^{-\beta}}dt < \infty.$$

then $\mathscr{H}^b_{\Phi,\Omega}$ *is a bounded operator from* $\dot{K}^{\alpha_1,p,q}_{v,\omega}(\mathbb{R}^n)$ *to* $\dot{K}^{\alpha_2,p,q}_{v,\omega}(\mathbb{R}^n)$.

Theorem 2.6. *Let* $0 < p < \infty, 1 \leq q < \infty$, $\Omega \in L^{q'}(S_{n-1})$, $\lambda > 0$ *and* $b \in Lip^{\beta}(\mathbb{R}^n)$ *với* $0 < \beta \leq 1$. *Suppose* $v, \omega \in \mathscr{W}_{\gamma}$, $\gamma > -n$ *and* $\omega(x') \geq c > 0$ *for all* $x' \in S_{n-1}$. *If* $\alpha_1 = \alpha_2 + \frac{n\beta}{n+\gamma}$ *and*

$$\mathscr{C}_6 = \int_0^{\infty} \frac{|\Phi(t)|}{t^{1-\frac{\gamma}{q}-\frac{n}{q}+(\lambda-\alpha_1)\left(1+\frac{\gamma}{n}\right)}(1+t^{-1})^{-\beta}}dt < \infty.$$

then $\mathscr{H}^b_{\Phi,\Omega}$ *is a bounded operator from* $M\dot{K}^{\alpha_1,\lambda,p,q}_{v,\omega}(\mathbb{R}^n)$ *to* $M\dot{K}^{\alpha_2,\lambda,p,q}_{v,\omega}(\mathbb{R}^n)$.

Notice that, from Theorems 2.5 and 2.6, by taking $\Phi(t) = t^{-n}\chi_{(1,\infty)}(t)$ and $\Omega \equiv 1$. We obtain an interesting result for the boundedness of commutators of the Hardy operators on two weighted Morrey–Herz spaces.

Chapter 3

TWO–WEIGHTED ESTIMATES FOR MULTILINEAR HAUSDORFF OPERATORS ON THE MORREY–HERZ SPACES

In this chapter, we establish some necessary and sufficient conditions for the boundedness of the multilinear Hausdorff operators $\mathscr{H}_{\Phi,\vec{A}}$ on the product of some two weighted function spaces such as the two-weighted Morrey, Herz and Morrey–Herz spaces. Moreover, some sufficient conditions for the boundedness of multilinear Hausdorff operators $\mathscr{H}_{\Phi,\vec{A}}$ on such spaces with respect to the Muckenhoupt weights are also given.

This chapter is written based on the paper [2].

3.1. Introduction

The research problem is to give some necessary and sufficient conditions for the boundedness the multilinear Hausdorff operator $\mathscr{H}_{\Phi,\vec{A}}$ on two weighted Morrey-Herz type spaces.

3.2. The $\mathscr{H}_{\Phi,\vec{A}}$ operator and the power weights

In this section, we give some results for the boundedness of the operator $\mathscr{H}_{\Phi,\vec{A}}$ on two weighted central Morrey spaces (Theorem 3.1), two weighted Herz spaces (Theorem 3.2)and two weighted Morrey-Herz spaces (Theorem 3.3).

Theorem 3.1. *Let* $\Phi : \mathbb{R}^n \to [0,\infty)$ *và* $v(x) = |x|^{\beta}, \omega(x) = |x|^{\gamma}$, $v_i(x) = |x|^{\beta_i}, \omega_i(x) = |x|^{\gamma_i}$, *với mọi* $i = 1,...,m$. *If the following conditions hold*

$$\sum_{i=1}^{m} \frac{\beta_i}{q_i} = \frac{\beta}{q}, \sum_{i=1}^{m} \left(\frac{n+\beta_i}{n+\beta}\right) \lambda_i = \lambda, \text{ và } \sum_{i=1}^{m} \frac{\gamma_i}{q_i} = \frac{\gamma}{q}$$

then $\mathscr{H}_{\Phi,\vec{A}}$ is bounded from $\prod_{i=1}^m \dot{M}^{\lambda_i,q_i}_{v_i,\omega_i}(\mathbb{R}^n)$ to $\dot{M}^{\lambda,q}_{v,\omega}(\mathbb{R}^n)$ if and only if

$$\mathscr{C}_7 = \int_{\mathbb{R}^n} \frac{\Phi(y)}{|y|^n} \prod_{i=1}^m \|A_i^{-1}(y)\|^{-(\beta_i+n)\lambda_i+(\gamma_i-\beta_i)\frac{1}{q_i}} dy < \infty.$$

Moreover, $\|\mathscr{H}_{\Phi,\vec{A}}\|_{\prod_{i=1}^m \dot{M}^{\lambda_i,q_i}_{v_i,\omega_i}(\mathbb{R}^n)\to \dot{M}^{\lambda,q}_{v,\omega}(\mathbb{R}^n)} \simeq \mathscr{C}_7$.

Theorem 3.2. *Let $\Phi : \mathbb{R}^n \to [0,\infty)$ and $v(x) = |x|^\beta, \omega(x) = |x|^\gamma$, $v_i(x) = |x|^{\beta_i}, \omega_i(x) = |x|^{\gamma_i}$, với mọi $i = 1, ..., m$. If the following conditions hold*

$$\sum_{i=1}^m \frac{\gamma_i}{q_i} = \frac{\gamma}{q}, \text{ và } \sum_{i=1}^m \left(1+\frac{\beta_i}{n}\right)\alpha_i = \left(1+\frac{\beta}{n}\right)\alpha,$$

then $\mathscr{H}_{\Phi,\vec{A}}$ is bounded from $\prod_{i=1}^m \dot{K}^{\alpha_i,p_i,q_i}_{v_i,\omega_i}(\mathbb{R}^n)$ to $\dot{K}^{\alpha,p,q}_{v,\omega}(\mathbb{R}^n)$ if and only if

$$\mathscr{C}_8 = \int_{\mathbb{R}^n} \frac{\Phi(y)}{|y|^n} \prod_{i=1}^m \|A_i^{-1}(y)\|^{\left(1+\frac{\beta_i}{n}\right)\alpha_i+\frac{n+\gamma_i}{q_i}} dy < \infty.$$

Moreover, $\|\mathscr{H}_{\Phi,\vec{A}}\|_{\prod_{i=1}^m \dot{K}^{\alpha_i,p_i,q_i}_{v_i,\omega_i}(\mathbb{R}^n)\to \dot{K}^{\alpha,p,q}_{v,\omega}(\mathbb{R}^n)} \simeq \mathscr{C}_8$.

Theorem 3.3. *Let $\Phi : \mathbb{R}^n \to [0,\infty)$, $\lambda, \lambda_i > 0$ and $v(x) = |x|^\beta, \omega(x) = |x|^\gamma$, $v_i(x) = |x|^{\beta_i}, \omega_i(x) = |x|^{\gamma_i}$, với mọi $i = 1, ..., m$. If the following conditions hold*

$$\sum_{i=1}^m \left(1+\frac{\beta_i}{n}\right)\lambda_i = \left(1+\frac{\beta}{n}\right)\lambda, \sum_{i=1}^m \frac{\gamma_i}{q_i} = \frac{\gamma}{q}, \text{ và } \sum_{i=1}^m \left(1+\frac{\beta_i}{n}\right)\alpha_i = \left(1+\frac{\beta}{n}\right)\alpha,$$

then we have that the operator $\mathscr{H}_{\Phi,\vec{A}}$ is bounded from $\prod_{i=1}^m M\dot{K}^{\alpha_i,\lambda_i,p_i,q_i}_{v_i,\omega_i}(\mathbb{R}^n)$ to $M\dot{K}^{\alpha,\lambda,p,q}_{v,\omega}(\mathbb{R}^n)$ if and only if

$$\mathscr{C}_9 = \int_{\mathbb{R}^n} \frac{\Phi(y)}{|y|^n} \prod_{i=1}^m \|A_i^{-1}(y)\|^{\left(1+\frac{\beta_i}{n}\right)(\alpha_i-\lambda_i)+\frac{n+\gamma_i}{q_i}} dy < \infty.$$

Moreover, $\|\mathscr{H}_{\Phi,\vec{A}}\|_{\prod_{i=1}^m M\dot{K}^{\alpha_i,\lambda_i,p_i,q_i}_{v_i,\omega_i}(\mathbb{R}^n)\to M\dot{K}^{\alpha,\lambda,p,q}_{v,\omega}(\mathbb{R}^n)} \simeq \mathscr{C}_9$.

Let us take the matrices $A_i(y) = \mathrm{diag}[s_i(y), ..., s_i(y)]$, where $s_1(y), ..., s_m(y) \neq 0$ almost everywhere in $\mathbb{R}^n$ for all $i = 1, ..., m$. we also obtain the necessary and sufficient conditions for the boundedness of the multilinear operator $\mathscr{H}_{\phi,\vec{s}}$ on two weighted central Morrey spaces (Corollary 3.1), two weighted Herz spaces (Corollary 3.2), two weighted Morrey-Herz spaces (Corollary 3.3).

Corollary 3.1. *Let ϕ be a nonnegative function and $v(x)=|x|^{\beta}, \omega(x)=|x|^{\gamma}$, $v_i(x)=|x|^{\beta_i}, \omega_i(x)=|x|^{\gamma_i}$, với mọi $i=1,...,m$. If the following conditions hold*

$$\sum_{i=1}^{m}\frac{\beta_i}{q_i}=\frac{\beta}{q},\ \sum_{i=1}^{m}\left(\frac{n+\beta_i}{n+\beta}\right)\lambda_i=\lambda,\ và\ \sum_{i=1}^{m}\frac{\gamma_i}{q_i}=\frac{\gamma}{q}$$

then $\mathscr{H}_{\phi,\vec{s}}$ is bounded from $\prod_{i=1}^{m}\dot{M}^{\lambda_i,q_i}_{v_i,\omega_i}(\mathbb{R}^n)$ to $\dot{M}^{\lambda,q}_{v,\omega}(\mathbb{R}^n)$ if and only if

$$\mathscr{C}_{7.1}=\int_{\mathbb{R}^n}\left(\prod_{i=1}^{m}|s_i(y)|^{(\beta_i+n)\lambda_i+(\beta_i-\gamma_i)\frac{1}{q_i}}\right)\phi(y)dy<\infty.$$

Moreover, $\|\mathscr{H}_{\phi,\vec{s}}\|_{\prod_{i=1}^{m}\dot{M}^{\lambda_i,q_i}_{v_i,\omega_i}(\mathbb{R}^n)\to\dot{M}^{\lambda,q}_{v,\omega}(\mathbb{R}^n)}\simeq\mathscr{C}_{7.1}$.

Corollary 3.2. *Let ϕ be a nonnegative function and $v(x)=|x|^{\beta}, \omega(x)=|x|^{\gamma}$, $v_i(x)=|x|^{\beta_i}, \omega_i(x)=|x|^{\gamma_i}$, với mọi $i=1,...,m$. If the following conditions hold*

$$\sum_{i=1}^{m}\frac{\gamma_i}{q_i}=\frac{\gamma}{q},\ và\ \sum_{i=1}^{m}\left(1+\frac{\beta_i}{n}\right)\alpha_i=\left(1+\frac{\beta}{n}\right)\alpha,$$

then $\mathscr{H}_{\phi,\vec{s}}$ is bounded from $\prod_{i=1}^{m}\dot{K}^{\alpha_i,p_i,q_i}_{v_i,\omega_i}(\mathbb{R}^n)$ to $\dot{K}^{\alpha,p,q}_{v,\omega}(\mathbb{R}^n)$ if and only if

$$\mathscr{C}_{8.1}=\int_{\mathbb{R}^n}\left(\prod_{i=1}^{m}|s_i(y)|^{-\left(1+\frac{\beta_i}{n}\right)\alpha_i-\frac{(n+\gamma_i)}{q_i}}\right)\phi(y)dy<\infty.$$

Moreover, $\|\mathscr{H}_{\phi,\vec{s}}\|_{\prod_{i=1}^{m}\dot{K}^{\alpha_i,p_i,q_i}_{v_i,\omega_i}(\mathbb{R}^n)\to\dot{K}^{\alpha,p,q}_{v,\omega}(\mathbb{R}^n)}\simeq\mathscr{C}_{8.1}$.

Corollary 3.3. *Let ϕ be a nonnegative function and, $\lambda,\lambda_i>0$ và $v(x)=|x|^{\beta}, \omega(x)=|x|^{\gamma}$, $v_i(x)=|x|^{\beta_i}, \omega_i(x)=|x|^{\gamma_i}$, với mọi $i=1,...,m$. If the following conditions hold*

$$\sum_{i=1}^{m}\left(1+\frac{\beta_i}{n}\right)\lambda_i=\left(1+\frac{\beta}{n}\right)\lambda,\ \sum_{i=1}^{m}\frac{\gamma_i}{q_i}=\frac{\gamma}{q},\ và\ \sum_{i=1}^{m}\left(1+\frac{\beta_i}{n}\right)\alpha_i=\left(1+\frac{\beta}{n}\right)\alpha,$$

then $\mathscr{H}_{\phi,\vec{s}}$ is bounded from $\prod_{i=1}^{m}M\dot{K}^{\alpha_i,\lambda_i,p_i,q_i}_{v_i,\omega_i}(\mathbb{R}^n)$ to $M\dot{K}^{\alpha,\lambda,p,q}_{v,\omega}(\mathbb{R}^n)$ if and only if

$$\mathscr{C}_{9.1}=\int_{\mathbb{R}^n}\left(\prod_{i=1}^{m}|s_i(y)|^{(\lambda_i-\alpha_i)\left(1+\frac{\beta_i}{n}\right)-\frac{(n+\gamma_i)}{q_i}}\right)\phi(y)dy<\infty.$$

Moreover, $\|\mathscr{H}_{\phi,\vec{s}}\|_{\prod_{i=1}^{m}M\dot{K}^{\alpha_i,p_i,q_i}_{v_i,\omega_i}(\mathbb{R}^n)\to M\dot{K}^{\alpha,p,q}_{v,\omega}(\mathbb{R}^n)}\simeq\mathscr{C}_{9.1}$.

By virtue of Corollary 3.1 one can claim that the weighted multilinear Hardy–Cesàro $U^{m,n}_{\psi,\vec{s}}$ is bounded from $\prod_{i=1}^m \dot{M}^{\lambda_i,q_i}_{v_i,\omega_i}(\mathbb{R}^n)$ to $\dot{M}^{\lambda,q}_{v,\omega}(\mathbb{R}^n)$ if and only if

$$\mathscr{C}_{7.2} = \int_{[0,1]^n} \left(\prod_{i=1}^m |s_i(t)|^{(\beta_i+n)\lambda_i+(\beta_i-\gamma_i)\frac{1}{q_i}} \right) \psi(t)dt < \infty.$$

Moreover, $\|U^{m,n}_{\psi,\vec{s}}\|_{\prod_{i=1}^m \dot{M}^{\lambda_i,q_i}_{v_i,\omega_i}(\mathbb{R}^n)\to \dot{M}^{\lambda,q}_{v,\omega}(\mathbb{R}^n)} \simeq \mathscr{C}_{7.2}$.

By virtue of Corollary 3.2 one can claim that the weighted multilinear Hardy–Cesàro$U^{m,n}_{\psi,\vec{s}}$ is bounded from $\prod_{i=1}^m \dot{K}^{\alpha_i,p_i,q_i}_{v_i,\omega_i}(\mathbb{R}^n)$ to $\dot{K}^{\alpha,p,q}_{v,\omega}(\mathbb{R}^n)$ if and only if

$$\mathscr{C}_{8.2} = \int_{[0,1]^n} \left(\prod_{i=1}^m |s_i(t)|^{-\left(1+\frac{\beta_i}{n}\right)\alpha_i-\frac{(n+\gamma_i)}{q_i}} \right) \psi(t)dt < \infty.$$

Moreover, $\|U^{m,n}_{\psi,\vec{s}}\|_{\prod_{i=1}^m \dot{K}^{\alpha_i,p_i,q_i}_{v_i,\omega_i}(\mathbb{R}^n)\to \dot{K}^{\alpha,p,q}_{v,\omega}(\mathbb{R}^n)} \simeq \mathscr{C}_{8.2}$.

Thus, Corollary 3.2 extends the results of Theorem 3.2 of N. M Chuong, N. T Hong, H. D Hung (2017).

3.3. The $\mathscr{H}_{\Phi,\vec{A}}$ operator and Muckenhoupt weights

In this section, we will address some sufficient conditions for the boundedness of the operator $\mathscr{H}_{\Phi,\vec{A}}$ on two-weighted space: central Morrey space (Theorem 3.4) and Morrey—Herz spaces (Theorem 3.5) associated with the class of the Muckenhoupt weights.

Theorem 3.4. *Let $1 \le q^*, \xi, \eta < \infty, -\frac{1}{q_i} < \lambda_i < 0$, for all $i = 1, ..., m$ and $v \in A_\eta, \omega \in A_\xi$ with the finite critical index r_v, r_ω for the reverse Hölder condition such that $\omega(B(0,R)) \lesssim v(B(0,R))$ for all $R > 0$. Assume that $q > q^*\xi r'_\omega, \delta_1 \in (1, r_\omega), \delta_2 \in (1, r_v), \lambda^* = \lambda_1 + \cdots + \lambda_m$ and*

$$\mathscr{C}_{10} = \int_{\mathbb{R}^n} \frac{|\Phi(y)|}{|y|^n} \prod_{i=1}^m |\det A_i^{-1}(y)|^{\frac{\xi}{q_i}} \|A_i(y)\|^{\frac{\xi n}{q_i}} \mathscr{A}_i(y)dy < \infty,$$

where

$$\mathscr{A}_i(y) = \left(\|A_i(y)\|^{n\left(\lambda_i+\frac{1}{q_i}\right)\frac{\delta_2-1}{\delta_2}} \chi_{\{y\in\mathbb{R}^n:\|A_i(y)\|\le 1\}} + \|A_i(y)\|^{n\eta\left(\lambda_i+\frac{1}{q_i}\right)} \chi_{\{y\in\mathbb{R}^n:\|A_i(y)\|>1\}} \right) \times$$
$$\times \left(\|A_i(y)\|^{-\frac{n}{q_i}\frac{\delta_1-1}{\delta_1}} \chi_{\{y\in\mathbb{R}^n:\|A_i(y)\|>1\}} + \|A_i(y)\|^{-\frac{\xi n}{q_i}} \chi_{\{y\in\mathbb{R}^n:\|A_i(y)\|\le 1\}} \right).$$

Then, $\mathscr{H}_{\Phi,\vec{A}}$ is bounded from $\prod_{i=1}^m \dot{M}^{\lambda_i,q_i}_{v,\omega}(\mathbb{R}^n)$ to $\dot{M}^{\lambda^*,q^*}_{v,\omega}(\mathbb{R}^n)$.

Theorem 3.5. *Let $1 \le q^*, \xi, \eta < \infty, \alpha_i < 0,\ \lambda_i \ge 0$, for all $i = 1, \dots, m$ and $\omega \in A_\xi, \nu \in A_\eta$ with the finite critical index r_ω, r_ν for the reverse Hölder condition such that $\omega(B_k) \lesssim \nu(B_k)$, với mọi $k \in \mathbb{Z}$. Assume that $q > \max\{mq^*, q^*\xi r'_\omega\}, \delta_1 \in (1, r_\omega), \delta_2 \in (1, r_\nu)$ và α^*, λ^* are two real numbers such that*

$$\lambda^* = \lambda_1 + \cdots + \lambda_m \text{ and } \frac{1}{m}\left(\frac{\alpha^*}{n} + \frac{1}{q^*}\right) = \frac{\alpha_i}{n} + \frac{1}{q_i} \text{ for all } i = 1, \dots, m.$$

If $\frac{\alpha^*}{n} + \frac{1}{q^*} \le 0$ and

$$\mathscr{C}_{11.1} = \prod_{i=1}^m \left(\int_{\mathbb{R}^n} \frac{|\Phi(y)|}{|y|^n} |\det A_i^{-1}(y)|^{\frac{m\xi}{q_i}} \|A_i(y)\|^{\frac{m\xi n}{q_i}} \mathscr{B}_{1i}(y) dy \right)^{\frac{1}{m}} < \infty,$$

where

$$\mathscr{B}_{1i}(y) = \|A_i(y)\|^{-\left(\left(\frac{n}{q^*}+\alpha^*\right)\frac{\delta_1-1}{\delta_1}+(\alpha^*-m\lambda_i)\frac{\delta_2-1}{\delta_2}-\xi\alpha^*\right)} \chi_{\{y\in\mathbb{R}^n:\|A_i(y)\|<1\}} + \\ + \|A_i(y)\|^{-\left(\left(\frac{n}{q^*}+\alpha^*\right)\xi+\eta(\alpha^*-m\lambda_i)-\alpha^*\frac{\delta_1-1}{\delta_1}\right)} \chi_{\{y\in\mathbb{R}^n:\|A_i(y)\|\ge1\}},$$

or $\frac{\alpha^*}{n} + \frac{1}{q^*} > 0$ và

$$\mathscr{C}_{11.2} = \prod_{i=1}^m \left(\int_{\mathbb{R}^n} \frac{|\Phi(y)|}{|y|^n} |\det A_i^{-1}(y)|^{\frac{m\xi}{q_i}} \|A_i(y)\|^{\frac{m\xi n}{q_i}} \mathscr{B}_{2i}(y) dy \right)^{\frac{1}{m}} < \infty,$$

where

$$\mathscr{B}_{2i}(y) = \|A_i(y)\|^{-\left(\frac{\xi n}{q^*}+(\alpha^*-m\lambda_i)\frac{\delta_2-1}{\delta_2}\right)} \chi_{\{y\in\mathbb{R}^n:\|A_i(y)\|<1\}} + \\ + \|A_i(y)\|^{-\left(\frac{n}{q^*}\frac{\delta_1-1}{\delta_1}+\eta(\alpha^*-m\lambda_i)\right)} \chi_{\{y\in\mathbb{R}^n:\|A_i(y)\|\ge1\}},$$

then we have $\mathscr{H}_{\Phi,\vec{A}}$ is bounded from $\prod_{i=1}^m M\dot{K}^{\alpha_i,\lambda_i,p_i,q_i}_{\nu,\omega}(\mathbb{R}^n)$ to $M\dot{K}^{\alpha^*,\lambda^*,p,q^*}_{\nu,\omega}(\mathbb{R}^n)$.

Theorem 3.6. *Let $1 \le q^*, \xi < \infty, \alpha_i < 0,\ \lambda_i \ge 0$, for all $i = 1, \dots, m$ and $\omega \in A_\xi$ with the finite critical index r_ω for the reverse Hölder condition such that. Assume that $q > q^*\xi r'_\omega, \delta \in (1, r_\omega)$ and α^*, λ^* are two real numbers satisfying*

$$\lambda^* = \lambda_1 + \cdots + \lambda_m \text{ và } \frac{\alpha^*}{n} + \frac{1}{q^*} = \frac{\sum_{i=1}^m \alpha_i}{n} + \frac{1}{q}.$$

If $\frac{\alpha_i}{n} + \frac{1}{q_i} \le 0$, for all $i = 1, \dots, m$ and

$$\mathscr{C}_{12.1} = \prod_{i=1}^m \left(\int_{\mathbb{R}^n} \frac{|\Phi(y)|}{|y|^n} |\det A_i^{-1}(y)|^{\frac{m\xi}{q_i}} \|A_i(y)\|^{\frac{m\xi n}{q_i}} \Psi_{1i}(y) dy \right)^{\frac{1}{m}} < \infty,$$

where

$$\Psi_{1i}(y) = \|A_i(y)\|^{m\left(\lambda_i - n\left(\frac{\alpha_i}{n}+\frac{1}{q_i}\right)\left(\frac{\delta-1}{\delta}\right)\right)}\chi_{\{y\in\mathbb{R}^n:\|A_i(y)\|<1\}} + \\ + \|A_i(y)\|^{m\xi\left(\lambda_i - n\left(\frac{\alpha_i}{n}+\frac{1}{q_i}\right)\right)}\chi_{\{y\in\mathbb{R}^n:\|A_i(y)\|\geq 1\}},$$

or $\frac{\alpha_i}{n}+\frac{1}{q_i} > 0$*, for all* $i = 1, \ldots, m$ *and*

$$\mathscr{C}_{12.2} = \prod_{i=1}^{m}\left(\int_{\mathbb{R}^n}\frac{|\Phi(y)|}{|y|^n}|\det A_i^{-1}(y)|^{\frac{m\xi}{q_i}}\|A_i(y)\|^{\frac{m\xi n}{q_i}}\Psi_{2i}(y)dy\right)^{\frac{1}{m}} < \infty,$$

where

$$\Psi_{2i}(y) = \|A_i(y)\|^{m\left(\lambda_i\left(\frac{\delta-1}{\delta}\right) - n\xi\left(\frac{\alpha_i}{n}+\frac{1}{q_i}\right)\right)}\chi_{\{y\in\mathbb{R}^n:\|A_i(y)\|<1\}} + \\ + \|A_i(y)\|^{m\left(\lambda_i\xi - n\left(\frac{\alpha_i}{n}+\frac{1}{q_i}\right)\left(\frac{\delta-1}{\delta}\right)\right)}\chi_{\{y\in\mathbb{R}^n:\|A_i(y)\|\geq 1\}},$$

then $\mathscr{H}_{\Phi,\vec{A}}$ *is bounded from* $\prod_{i=1}^m M\dot{K}_{\omega}^{\alpha_i,\lambda_i,p_i,q_i}(\mathbb{R}^n)$ *to* $M\dot{K}_{\omega}^{\alpha^*,\lambda^*,p,q^*}(\mathbb{R}^n)$.

As consequences of Theorems 3.5 and 3.6, by letting $\lambda_1 = \cdots = \lambda_m = 0$ we also obtain the sufficient conditions for the boundedness of the multilinear Hausdorff operators $\mathscr{H}_{\Phi,\vec{A}}$ on two-weighted Herz spaces with the Muckenhoupt weights.

Chapter 4

WEIGHTED ESTIMATES FOR COMMUTATORS OF HAUSDORFF OPERATORS ON THE HEISENBERG GROUP

In this chapter, we give some sufficient conditions for the boundedness of commutators $\mathscr{H}^b_{\Phi,\Omega}$ and $\mathscr{H}^b_{\Phi,A}$ of Hausdorff operators with symbols in weighted ℓ-central BMO type spaces on the Herz spaces, central Morrey spaces and Morrey–Herz spaces associated with both power weights and Mukenhoupt weights on the Heisenberg group.

This chapter is written based on the paper [3].

4.1. Giới thiệu

The research problem is to study some sufficient conditions for the boundedness of the commutator $\mathscr{H}^b_{\Phi,\Omega}$ and the commutator $\mathscr{H}^b_{\Phi,A}$ on the Heisenberg group.

4.2. The commutator $\mathscr{H}^b_{\Phi,\Omega}$ and the power weights

In this section, we give sufficient conditions for the boundedness of commutator $\mathscr{H}^b_{\Phi,\Omega}$ on power weighted central Morrey spaces (Theorem 4.1), the weighted Morrey-Herz spaces (Theorem 4.2).

Theorem 4.1. *Let $1 \leq q < \infty$, $1 < q_1, r_1 < \infty$ and $\omega(x) = |x|_h^\gamma$, $\gamma > -Q$. Assume that $\Omega \in L^{q'}(S_{Q-1})$, $b \in C\dot{M}O^{\ell,r_1}_\omega(\mathbb{H}^n), \ell < \frac{1}{Q}$ and $\lambda \in (-\frac{1}{q}, 0)$, $\lambda_1 \in (-\frac{1}{q_1}, 0)$, $\lambda_1 = \lambda - \ell$. If $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{r_1}$ and*

$$\mathscr{C}_{13} = \int_0^\infty \frac{\Phi(t)}{t^{1+(Q+\gamma)\lambda_1}} \left(1 + \Psi(t) + t^{-(Q+\gamma)\ell}\right) dt < \infty,$$

then $\mathscr{H}^b_{\Phi,\Omega}$ is bounded from $\dot{M}^{\lambda_1,q_1}_\omega(\mathbb{H}^n)$ to $\dot{M}^{\lambda,q}_\omega(\mathbb{H}^n)$.

Theorem 4.2. *Let $1 \leq p,q < \infty$, $1 < q_1, r_1 < \infty$ and $\omega(x) = |x|_h^{\gamma}$, $\gamma > -Q$. Assume that $\Omega \in L^{q'}(S_{Q-1})$, $b \in C\dot{M}O_{\omega}^{\ell,r_1}(\mathbb{H}^n), \ell < \frac{1}{Q}$, $\lambda \geq 0$ and $\alpha_1 = \alpha + (Q+\gamma)(\frac{1}{r_1} + \ell)$. If $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{r_1}$ and*

$$\mathscr{C}_{15} = \int_0^{\infty} \frac{\Phi(t)}{t^{1-\frac{Q+\gamma}{q_1}+\lambda-\alpha_1}} \left(1 + \Psi(t) + t^{-(Q+\gamma)\ell}\right) dt < \infty,$$

then $\mathscr{H}_{\Phi,\Omega}^b$ is bounded from $M\dot{K}_{\omega}^{\alpha_1,\lambda,p,q_1}(\mathbb{H}^n)$ to $M\dot{K}_{\omega}^{\alpha,\lambda,p,q}(\mathbb{H}^n)$.

4.3. Commutator $\mathscr{H}_{\Phi,\Omega}^b$ and Muckenhoupt weights

In this section, we have sufficient conditions for the boundedness of commutator $\mathscr{H}_{\Phi,\Omega}^b$ on central Morrey spaces with Muckenhoupt weights (Theorem 4.3).

Theorem 4.3. *Let $1 \leq q, q_1^*, r_1^*, \zeta < \infty$, $0 \leq \ell < \frac{1}{Q}$, $\omega \in A_{\zeta}$ with the finite critial index r_{ω} for the reverse Hölder condition. Assume that $\Omega \in L^{q'}(S_{Q-1})$, $b \in C\dot{M}O_{\omega}^{r_1^*,\ell}(\mathbb{H}^n), \delta \in (1, r_{\omega})$, $\lambda \in (-\frac{1}{q}, 0)$, $\lambda_1 \in (-\frac{1}{q_1}, 0)$ and $\lambda_1 = \lambda - \ell$. If $\frac{1}{q} > \left(\frac{1}{q_1^*} + \frac{1}{r_1^*}\right) \zeta \frac{r_{\omega}}{r_{\omega}-1}$ and*

$$\mathscr{C}_{14} = \int_0^{\infty} \frac{\Phi(t)}{t} \left(t^{-Q\frac{(\delta-1)\lambda_1}{\delta}} \chi_{(0,1]}(t) + t^{-Q\zeta\lambda_1} \chi_{(1,\infty)}(t)\right) \left(1 + \Psi_1(t)\right) dt < \infty,$$

then $\mathscr{H}_{\Phi,\Omega}^b$ is bounded from $\dot{M}_{\omega}^{\lambda_1,q_1^*}(\mathbb{H}^n)$ to $\dot{M}_{\omega}^{\lambda,q}(\mathbb{H}^n)$.

4.4. Commutator $\mathscr{H}_{\Phi,A}^b$ and power weights

In this section, we give sufficient conditions for the boundedness of commutator $\mathscr{H}_{\Phi,A}^b$ on central Morrey space (Theorem 4.4), Morrey-Herz space (Theorem 4.5) with power weights.

Theorem 4.4. *Let $1 \leq q < \infty$, $1 < q_1, r_1 < \infty$, $\gamma > -Q$, $\omega(x) = |x|_h^{\gamma}$, $b \in C\dot{M}O_{\omega}^{r_1}(\mathbb{H}^n)$ and $\lambda \in (-\frac{1}{q_1}, 0)$. If $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{r_1}$ and*

$$\mathscr{C}_{16} = \int_{\mathbb{H}^n} \frac{\Phi(y)}{|y|_h^Q} . \psi(y) . \mu(y) \|A(y)\|^{(Q+\gamma)(\frac{1}{q_1}+\lambda)} dy < \infty,$$

then $\mathscr{H}_{\Phi,A}^b$ is bounded from $\dot{M}_{\omega}^{\lambda,q_1}(\mathbb{H}^n)$ to $\dot{M}_{\omega}^{\lambda,q}(\mathbb{H}^n)$.

Theorem 4.5. *Let $1 \le p, q < \infty$, $1 < q_1, r_1 < \infty$, $\gamma > -Q$, $\omega(x) = |x|_h^\gamma$, $b \in C\dot{M}O_\omega^{r_1}(\mathbb{H}^n)$, $\lambda \ge 0$, $\alpha_2 = \alpha + \frac{Q+\gamma}{r_1}$. If $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{r_1}$ and*

$$\mathscr{C}_{18} = \int_{\mathbb{H}^n} \frac{\Phi(y)}{|y|_h^Q}.\psi(y).\mu(y)(2-\kappa^*)^{1-\frac{1}{p}} \|A(y)\|^{\lambda-\alpha_2} \Big(\sum_{i=\kappa^*-1}^{0} 2^{i(\lambda-\alpha_2)} \Big) dy < \infty,$$

with $\kappa^* = \kappa^*(y)$ is the greatest integer number such that

$$\|A(y)\|.\|A^{-1}(y)\| < 2^{-\kappa^*}, \text{ với mọi hầu khắp } y \in \mathbb{H}^n,$$

then $\mathscr{H}_{\Phi,A}^b$ is bounded from $M\dot{K}_\omega^{\alpha_2,\lambda,p,q_1}(\mathbb{H}^n)$ to $M\dot{K}_\omega^{\alpha,\lambda,p,q}(\mathbb{H}^n)$.

4.5. Commutator $\mathscr{H}_{\Phi,A}^b$ and Muckenhoupt weights

In this section, we establish sufficient conditions for the boundedness of commutator $\mathscr{H}_{\Phi,A}^b$ on central Morrey spaces with Muckenhoupt weights (Theorem 4.6).

Theorem 4.6. *Let $1 \le q, q_1^*, r_1^*, \zeta < \infty$, $\omega \in A_\zeta$ with the finite critial index r_ω for the reverse Hölder condition, $b \in C\dot{M}O_\omega^{r_1^*}(\mathbb{H}^n)$, $\lambda \in (-\frac{1}{q_1^*}, 0)$ and $\delta \in (1, r_\omega)$. If $\frac{1}{q} > \left(\frac{1}{q_1^*} + \frac{1}{r_1^*} \right) \zeta \frac{r_\omega}{r_\omega - 1}$ and*

$$\begin{aligned} \mathscr{C}_{17} &= \int_{\mathbb{R}^n} \frac{\Phi(y)}{|y|_h^Q}.\psi_1(y).\mu_1(y) \times \\ &\times \big(\|A(y)\|^{Q\zeta\lambda} \chi_{\{y \in \mathbb{H}^n : \|A(y)\| \le 1\}} + \|A(y)\|^{Q\frac{(\delta-1)\lambda}{\delta}} \chi_{\{y \in \mathbb{H}^n : \|A(y)\| > 1\}} \big) dy < \infty, \end{aligned}$$

then $\mathscr{H}_{\Phi,A}^b$ is bounded from $\dot{M}_\omega^{\lambda,q_1^*}(\mathbb{H}^n)$ to $\dot{M}_\omega^{\lambda,q}(\mathbb{H}^n)$.

CONCLUSION

1. Results

New results presented in the thesis are as follows:

1) We give necessary and sufficient conditions for the boundedness of rough Hausdorff operators $\mathscr{H}_{\Phi,\Omega}$ on Herz, Morrey and Morrey–Herz spaces with absolutely homogeneous weights. Especially, the estimates for operator norms in each case are worked out. Moreover, we also establish the boundedness of the commutators of rough Hausdorff operators $\mathscr{H}^b_{\Phi,\Omega}$ on the two weighted Morrey–Herz type spaces with their symbols belonging to Lipschitz space.

2) We establish some necessary and sufficient conditions for the boundedness of the multilinear Hausdorff operators $\mathscr{H}_{\Phi,\vec{A}}$ on the product of some two weighted function spaces such as the two-weighted Morrey, Herz and Morrey–Herz spaces. Moreover, some sufficient conditions for the boundedness of multilinear Hausdorff operators $\mathscr{H}_{\Phi,\vec{A}}$ on such spaces with respect to the Muckenhoupt weights are also given.

3) We give some sufficient conditions for the boundedness of commutators $\mathscr{H}^b_{\Phi,\Omega}$ and $\mathscr{H}^b_{\Phi,A}$ of Hausdorff operators with symbols in weighted ℓ-central BMO type spaces on the Herz spaces, central Morrey spaces and Morrey–Herz spaces associated with both power weights and Mukenhoupt weights on the Heisenberg group.

2. Recommendation

Some open problems can be further studied. More precisely, there are the following problems.

1) We will study the norm estimate of the operator $\mathscr{H}_{\Phi,\Omega}$ and the commutator $\mathscr{H}^{b}_{\Phi,\Omega}$ with the symbol belonging Lipschitz space, on the Morrey-Herz type spaces with homogeneous weights. We will find a relationship between the singular integral operator and the Hausdorff operator.

2) We will study norm estimate of the multilinear Hausdorff operator commutator $\mathscr{H}_{\Phi,\vec{A}}$, with the symbol belonging Lipschitz space on the product of the type Morrey-Herz spaces with two Muckenhoupt weights.

3) We will study norm estimate of some Hausdorff operator on Heisenberg group on Morrey-Herz type spaces with two Muckenhoupt weights.

AUTHOR'S WORKS RELATED TO THE THESIS THAT HAVE BEEN PUBLISHED

[1] N. M. Chuong, D. V. Duong, N. D. Duyet, (2020), Weighted Morrey-Herz space estimates for rough Hausdorff operator and its commutators, *J. Pseudo-Differ. Oper. Appl.* Vol. 11, No. 2, 753–787. (SCIE)

[2] N. M. Chuong, D. V. Duong, N. D. Duyet, (2020), Two Weighted estimates for multilinear Hausdorff Operators on the Morrey-Herz Spaces, *Adv. Oper. Theory.* Vol. 5, No. 4, 1780–1813. (ESCI/Scopus)

[3] N. M. Chuong, D. V. Duong, N. D. Duyet, (2020), Weighted Estimates for Commutators of Hausdorff Operators on the Heisenberg Group, *Russian Mathematics.* Vol. 64, No. 2, pp. 35–55. (ESCI/Scopus)